Polynomial invariants of a semisimple and cosemisimple Hopf algebra based on braiding structures

Michihisa Wakui (Kansai Univ.)

August 31st, 2009
in La Falda, Córdoba

Contents

— **a new family of monoidal Morita invariants of a semisimple and cosemisimple Hopf algebra of finite dimension by using the braiding structures.**

— **computational results and usefulness of our invarinats for detecting whether representation categories of Hopf algebras are monoidally equivalent or not.**

— **basic properties of our invariants, including integer property and stability under extensions of the base field.**

Notation

- k = a field.
- A = a finite-dimensional Hopf algebra over k.
- ${}_A\mathbb{M}^{\text{f.d.}}$ = the monoidal category of finite-dimensional left A-modules.

Problem

For Hopf algebras A and B, when ${}_A\mathbb{M}$, ${}_B\mathbb{M}$ are equivalent as monoidal categories?

Known results Schauenburg has shown that

$$\mathbb{M}^A \simeq \mathbb{M}^B \text{ as monoidal categories}$$

$\iff$ there is an (A, B) bi-Galois extension of k

$\underset{\text{(f.d. case)}}{\iff}$ A and B are cocycle deformations of each other.

Masuoka, Doi, Takeuchi and Schauenburg have determined the bi-Galois objects and cocycle deformations for various special families of Hopf algebras.

Motivation (Quantum invariants of knots and 3-manifolds)

Given $M \in {}_A\mathbb{M}^{\text{f.d.}}$, we have a function

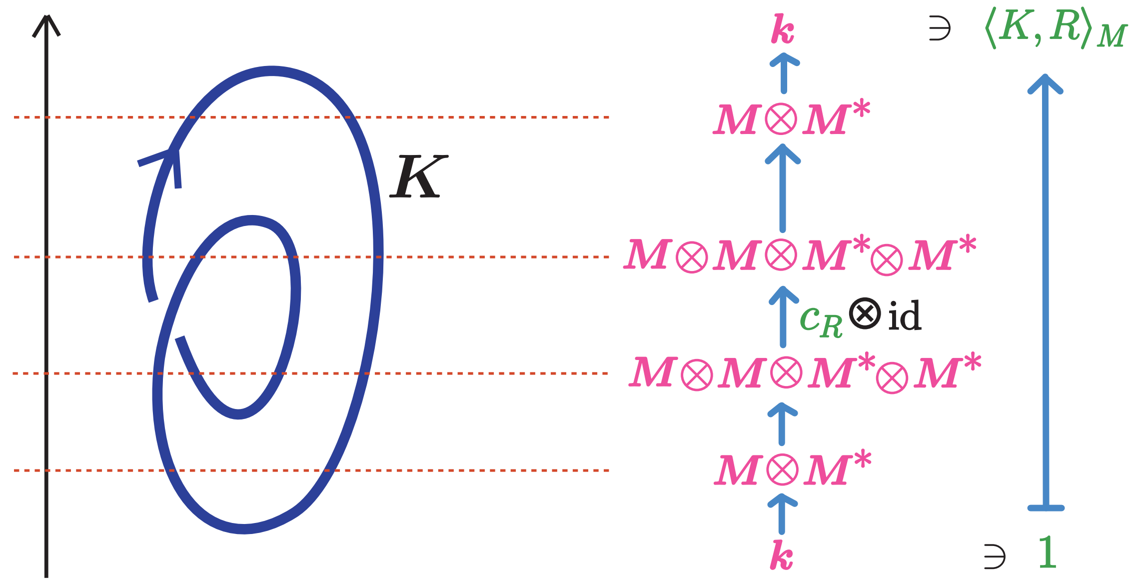
$$\left.\begin{array}{r} \text{a quasitriangular str. } R \in A \otimes A \\ \text{a framed knot } K \text{ in } \mathbb{S}^3 \end{array}\right\} \longmapsto \langle K, R\rangle_M \in \boldsymbol{k}.$$

For each fixed R and M, the function $\langle -, R\rangle_M$ is a topological invariant, so called a quantum invariant.

In this research we fix K, especially $K =$ **, and study on**

$\{\langle K, -\rangle_M\}_{M \in {}_A\mathbb{M}^{\text{f.d.}}}$ as an invariant of Hopf algebras.

Reshetikhin and Turaev's idea



Drinfeld element

If $R = \sum_i a_i \otimes b_i$ is a QT structure of A, then $u = \sum_i S(b_i)a_i \in A$ is invertible, and

$$S^2(a) = uau^{-1} \quad \text{for all } a \in A. \qquad (\dagger)$$

This element u is called the Drinfeld element associated to R. If A is semisimple and cosemisimple, then u belongs to the center of A by ($\dagger$) and $S^2 = \mathrm{id}_A$ [Etingof and Gelaki].

Braided dimension For $M \in {}_A\mathbb{M}^{\text{f.d.}}$ we define

$$\underline{\dim}_R M = \text{Trace}(\text{the left action of } u \text{ on } M),$$

and call it the R-dimension of M. The R-dimension is a special case of the braided dimension of M in the left rigid braided monoidal category $({}_A\mathbb{M}^{\text{f.d.}}, c_R)$.

Theorem (**Etingof and Gelaki**) **If A is cosemisimple, then**
(1) the set of QT structures $\underline{\mathrm{Braid}}(A)$ is finite,
(2) provided that A is semisimple, $(\dim M)1_k \neq 0$ for any absolutely simple left A-module M.

Definition of polynomial invariants

Suppose that A is semisimple and cosemisimple, and fix a positive integer d. Let $\{M_1, \ldots, M_t\}$ be a full set of non-isomorphic absolutely simple left A-modules of dimension d. Then we define

$$P_A^{(d)}(x) := \prod_{i=1}^{t} \prod_{R \in \underline{\mathrm{Braid}}(A)} \left(x - \frac{\dim_R M_i}{\dim M_i} \right) \quad \in \quad k[x].$$

Theorem **If A and B are monoidally Morita equivalent, then $P_A^{(d)}(x) = P_B^{(d)}(x)$ for all d.**

Satoshi Suzuki's Hopf algebras

Let $N \geq 1$ be an odd integer and $n \geq 2$ an integer. Let G_{Nn} be the finite group presented by

$$G_{Nn} = \left\langle\, h,\ t,\ w \;\middle|\; \begin{matrix} t^2 = h^{2N} = 1,\ w^n = h^N, \\ tw = w^{-1}t,\ ht = th,\ hw = wh \end{matrix} \right\rangle.$$

Suppose that $\mathrm{ch}(k) \neq 2$, and define Δ, ε, S by

$$\Delta(h) = h \otimes h, \qquad \varepsilon(h) = 1,\ S(h) = h^{-1},$$
$$\Delta(t) = h^N wt \otimes e_1 t + t \otimes e_0 t, \qquad \varepsilon(t) = 1,\ S(t) = (e_0 - e_1 w)t,$$
$$\Delta(w) = w \otimes e_0 w + w^{-1} \otimes e_1 w, \quad \varepsilon(w) = 1,\ S(w) = e_0 w^{-1} + e_1 w,$$

where $e_0 = \frac{1+h^N}{2}, e_1 = \frac{1-h^N}{2}$. Then, $A_{Nn} = (k[G_{Nn}], \Delta, \varepsilon, S)$ is a semisimple and cosemisimple Hopf algebra of dimension $4nN$, and self-dual i.e. $A_{Nn} \cong (A_{Nn})^*$. [A_{1n} is well-studied by Masuoka.]

Remark A_{12} is isomorphic to the Kac-Paljutkin algebra.

Computational results

Suppose that k contains a primitive $4nN$-th root of unity ω.
If $n \geq 3$ is odd, then $P^{(d)}_{A_{Nn}}(x) = P^{(d)}_{k[G_{Nn}]}(x)$ $(d = 1, 2)$.
However,

Theorem If n is even, then the pair of two Hopf algebras A_{Nn} and $k[G_{Nn}]$ gives an example of that their representation categories are not equivalent, meanwhile their representation rings are isomorphic.

(Proof) If $n = 2$, then

$$P^{(1)}_{A_{N2}}(x) = \prod_{s,i=0}^{N-1} (x - \omega^{-16is^2})^{16}(x + \omega^{-8is^2})^{8}(x + \omega^{-16is^2})^{8},$$

$$P^{(1)}_{k[G_{N2}]}(x) = \prod_{s,i=0}^{N-1} (x - \omega^{-16is^2})^{32}.$$

So, $P^{(1)}_{A_{N2}}(-1) = 0 \neq P^{(1)}_{k[G_{N2}]}(-1)$.

If $n \geq 4$ is even, then

$$P^{(2)}_{A_{Nn}}(x) = \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{n}{2}} \prod_{j=0}^{n-1} (x^2 - \omega^{-4i(2s+1)^2 n - 2(2j-1)(2t-1)^2 N}) \times \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{n-2}{2}} \prod_{j=0}^{n-1} (x - \omega^{-8is^2 n - 4(2j-1)t^2 N})^2,$$

$$P^{(2)}_{k[G_{Nn}]}(x) = \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{n}{2}} \prod_{j=0}^{n-1} (x^2 - \omega^{-4i(2s+1)^2 n - 4j(2t-1)^2 N}) \times \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{n-2}{2}} \prod_{j=0}^{n-1} (x - \omega^{-8is^2 n - 8j t^2 N})^2.$$

Thus $P^{(2)}_{A_{Nn}}(\omega^{-N}) = 0 \neq P^{(2)}_{k[G_{Nn}]}(\omega^{-N})$.

On the other hand, $\mathrm{K}_0(A_{Nn}), \mathrm{K}_0(k[G_{Nn}])$ **are isomorphic to**

$$\mathbb{Z}\left\langle \begin{matrix} a, b, c, \\ x_1, \ldots, x_{n-1} \end{matrix} \middle| \begin{matrix} a^2 = b^2 = c^N = 1, \ a x_i = x_i, \\ b x_i = x_{n-i}, \ x_i x_j = c^{\frac{1-(-1)^{ij}}{2}} (x_{|i-j|} + x_{i+j}) \end{matrix} \right\rangle^{\text{abel}},$$

where $x_0 = 1 + a, \ x_n = b(1+a), \ x_{n+i} = x_{n-i} \ (1 \leq i \leq n-1)$.

Integer property

Let K be an extension field of k. Then $A^K = A \otimes K$ becomes a Hopf algebra over K, and the group $\mathrm{Aut}(K/k)$ acts on $H := A^K$ and <u>Braid</u>(H) by

$$\begin{aligned} &\sigma(a \otimes c) = a \otimes \sigma(c), \\ &R^\sigma = \sum(\sigma^{-1}\alpha_i) \otimes_K (\sigma^{-1}\beta_i), \end{aligned} \quad \begin{pmatrix} \sigma \in \mathrm{Aut}(K/k),\ a \in A, \\ c \in K,\ R = \sum \alpha_i \otimes_K \beta_i \end{pmatrix}.$$

Theorem If K/k is a finite Galois extension, then $P^{(d)}_{A^K}(x) \in (k \cap Z_K)[x]$ for any d. Here,

$$Z_K = \begin{cases} \text{(the algebraic integers in } K) & (\mathrm{ch}(K) = 0), \\ \text{(the algebraic closure of } \mathbb{F}_p \text{ in } K) & (\mathrm{ch}(K) = p). \end{cases}$$

Corollary If K is a finite Galois extension field of $k = \mathbb{Q}$, then $P^{(d)}_{A^K}(x) \in \mathbb{Z}[x]$ for any d.

Stability under extensions of the base field

An extension of field K/k yields a natural injection $\underline{\text{Braid}}(A) \longrightarrow \underline{\text{Braid}}(A^K)$, which defines

$$R = \sum_i \alpha_i \otimes \beta_i \longmapsto R^K = \sum_i (\alpha_i \otimes 1_k) \otimes_K (\beta_i \otimes 1_k).$$

By using a result of Etingof and Gelaki one can show the existence of a separable finite extension field L of k such that $\underline{\text{Braid}}(A^L) = \underline{\text{Braid}}(A^E)$ for any extension E/L. This leads to the following theorem.

Theorem **There is a separable finite extension field L of k such that for any extension E/L and for any positive integer d, $P^{(d)}_{A^E}(x) = P^{(d)}_{A^L}(x)$ in $E[x]$.**

Reference

arXiv:0907.0089, M. Wakui, Polynomial invariants for a semisimple and cosemisimple Hopf algebra of finite dimension, to appear in JPAA.