Indecomposability of weak Hopf algebras

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Motivation

Chebel and Makhlouf's research [3]

$\Downarrow$
- **Kaplansky type construction for WBA,**
- **classification of WBAs over $\mathbb{C}$ of dim ≤ 3**

Direct sum construction (suggested by Masuoka)

$\Downarrow$

Indecomposability of WBAs

$\Downarrow$

Several questions

- **What are properties preserving under direct sum?**
- **Is any Hopf algebra indecomposable?**
- **Can it be interpreted by a categorical language?** (suggested by Shimizu)

[3] Z. Chebel and A. Makhlouf, "Kaplansky's construction type and classification of weak bialgebras and weak Hopf algebras", J. Generalized Lie Theory Appl. 9 (2015), no. S1, Art. ID S1-008, 9 pp.

Contents

Throughout this talk,

- k **is a field,**
- H **is an algebra and coalgebra over** k **with comultiplication** $\Delta = \Delta_H$ **and counit** $\varepsilon = \varepsilon_H$.
- **we use Sweedler's notation as** $\Delta(x) = x_{(1)} \otimes x_{(2)}$.
- $\Delta^{(2)} = (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.

§1. Weak Hopf algebras: Definitions and properties

Definition 1 (Böhm, Nill and Szlachányi [1])

(1) H is called a **weak bialgebra** (abb. **WBA**) over $\boldsymbol{k}$ if the following three conditions are satisfied:

(WH1) $\Delta(xy) = \Delta(x)\Delta(y)$ for $\forall\, x, y \in H$,

(WH2) $\Delta^{(2)}(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$,

(WH3) (1) $\varepsilon(xyz) = \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z)$,
(2) $\varepsilon(xyz) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z)$ for $\forall\, x, y \in H$.

(2) Let $S : H \longrightarrow H$ be a $\boldsymbol{k}$-linear transformation. The pair (H, S) is called a **weak Hopf algebra** (abb. **WHA**) over $\boldsymbol{k}$ if (WH1),(WH2),(WH3) and the following conditions are satisfied:

(WH4) (1) $x_{(1)}S(x_{(2)}) = \varepsilon(1_{(1)}x)1_{(2)}$,
(2) $S(x_{(1)})x_{(2)} = 1_{(1)}\varepsilon(x1_{(2)})$,
(3) $S(x_{(1)})x_{(2)}S(x_{(3)}) = S(x)$ for $\forall\, x \in H$.

[1] **G. Böhm, F. Nill, K. Szlachányi, "Weak Hopf algebras I. Integral theory and C^*-structure", J. Algebra 221 (1999), 385–438.**

Definition 1 (continued)

The above S is said to be an **antipode** of H or (H, S).

Remark.

1. a weak Hopf algebra = a quantum groupoid = a $\times_R$-bialgebra (introduced by Takeuchi [12]) in which R is separable (Schauenburg [9])
2. a face algebra (introduced by Hayashi [5]) = a weak Hopf algebra whose counital subalgebras are commutative.
3. a weak bialgebra is a bialgebra iff $\Delta(1) = 1 \otimes 1$.
4. Analogously in case of a bialgebra, an antipode for a weak bialgebra is unique if exists.

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[5] T. Hayashi, "Face algebras I. A generalization of quantum group theory", J. Math. Soc. Japan 50 (1998), 293–315.

[9] P. Schauenburg, "Weak Hopf algebras and quantum groupoids", Banach Center Publ. 61 (2003), 171–188.

[11] M. Takeuchi, "Groups of algebras over $A \otimes \bar{A}$", J. Math. Soc. Japan 29 (1977), 459–492.

Define $\varepsilon_t, \varepsilon_s$ by the RHSs of (WH4.1),(WH4.2):

$$\boldsymbol{\varepsilon_t(x) = \varepsilon(1_{(1)}x)1_{(2)},} \tag{1}$$

$$\boldsymbol{\varepsilon_s(x) = 1_{(1)}\varepsilon(x1_{(2)}).} \tag{2}$$

ε_t and ε_s are called the target and source counital maps, respectively.

Lemma 2

$\varepsilon_t,\ \varepsilon_s$ have the following properties:

(1) $\varepsilon_t^2 = \varepsilon_t,\ \varepsilon_s^2 = \varepsilon_s$.

(2) (i) $x_{(1)} \otimes \varepsilon_t(x_{(2)}) = 1_{(1)}x \otimes 1_{(2)}$,

(ii) $\varepsilon_s(x_{(1)}) \otimes x_{(2)} = 1_{(1)} \otimes x1_{(2)}$ for $\forall\, x \in H$.

In particular,

$$1_{(1)} \otimes \varepsilon_t(1_{(2)}) = 1_{(1)} \otimes 1_{(2)} = \varepsilon_s(1_{(1)}) \otimes 1_{(2)}.$$

(3) (i) $\varepsilon_t(x) = x \ \Leftrightarrow\ \Delta(x) = 1_{(1)}x \otimes 1_{(2)}$,

(ii) $\varepsilon_s(x) = x \ \Leftrightarrow\ \Delta(x) = 1_{(1)} \otimes x1_{(2)}$ for $\forall\, x \in H$.

(4) $x = \varepsilon_t(x_{(1)})x_{(2)} = x_{(1)}\varepsilon_s(x_{(2)})$.

Lemma 3

Set $H_t := \varepsilon_t(H),\ H_s := \varepsilon_s(H)$, which are called the **target and source subalgebras** of H, respectively. Then,

(1) actually, they are subalgebras of H,

(2) any elements in H_t and in H_s are commutative,

(3) $\Delta(1) \in H_s \otimes H_t$.

Definition 4

Let H_1 and H_2 be two bialgebras over $\boldsymbol{k}$. An algebra and coalgebra map $f : H_1 \longrightarrow H_2$ is called a **weak bialgebra map**. If H_1 and H_2 have antipodes S_1 and S_2, respectively, then a weak bialgebra map f satisfying $f \circ S_1 = S_2 \circ f$ is called a **weak Hopf algebra map**. A bijective weak bialgebra or Hopf algebra map is called an isomorphism.

As the same argument in Hopf algebra theory, one can define the dual H° for a weak bialgebra or Hopf algebra H:

$$H^\circ := \{\ p \in H^* \mid \dim(k[H]p) < \infty\ \}, \tag{3}$$

where H^* denotes the dual vector space of H, and

$$k[H] = \left\{ \sum_{x \in H} c_x x \;\middle|\; \begin{array}{l} c_x \in k,\ c_x = 0 \text{ except for} \\ \text{finitely many } x \in H \end{array} \right\},$$

and $\Big(\sum_{x \in H} c_x x\Big) p \in H^*$ is defined by

$$\Big(\Big(\sum_{x \in H} c_x x\Big) p\Big)(h) = \sum_{x \in H} c_x p(hx) \qquad (h \in H).$$

Proposition 5

The antipode S_{H° in the dual weak Hopf algebra H° is an anti-algebra and anti-coalgebra map.

In finite-dimensional case $H^\circ = H^*$, and the structure maps of the dual weak bialgebra $H^* = (H^*, \Delta_{H^*}, \varepsilon_{H^*})$ are given as follows: for all $x, y \in H$ and $p, q \in H^*$

- $(pq)(x) = p(x_{(1)})q(x_{(2)})$,
- $1_{H^*} = \varepsilon$ (= the counit of H),
- $\langle \Delta_{H^*}(p),\ x \otimes y \rangle = p(xy)$,
- $\varepsilon_{H^*}(p) = p(1)$.

If H is a weak Hopf algebra with antipode S, then H^* also has an antipode S_{H^*} defined by

- $\langle S_{H^*}(p),\ x \rangle = \langle p,\ S(x) \rangle$.

The usual k-linear isomorphism $\iota : H \longrightarrow H^{**} = (H^*)^*$ gives a weak Hopf algebra isomorphism.

Corollary 6 (Böhm, Nill and Szlachányi [1])

For any finite-dimensional weak Hopf algebra $H = (H, S)$, the antipode S is an anti-algebra and anti-coalgebra map.

§2. Indecomposable weak bialgebras

- For two algebras A and B over k, the direct sum $A \oplus B$ becomes an algebra with the following multiplication and identity element 1:

$$(a_1 + b_1)(a_2 + b_2) = a_1 a_2 + b_1 b_2,$$
$$1 = 1_A + 1_B,$$

where $a_1, a_2 \in A,\ b_1, b_2 \in B$, and $1_A, 1_B$ are the identity elements of A and B, respectively.

- For two coalgebras $C = (C, \Delta_C, \varepsilon_C)$ and $D = (D, \Delta_D, \varepsilon_D)$ over k, the direct sum $C \oplus D$ becomes a coalgebra with the following comultiplication Δ and counit ε:

$$\Delta(c + d) = \Delta_C(c) + \Delta_D(d),$$
$$\varepsilon(c + d) = \varepsilon_C(c) + \varepsilon_D(d) \quad \text{for } c \in C,\ d \in D.$$

Theorem 7 (Direct sum construction of WBAs)

Let $A = (A, \Delta_A, \varepsilon_A)$ and $B = (B, \Delta_B, \varepsilon_B)$ be two weak bialgebras over $\boldsymbol{k}$, and set $H = A \oplus B$. Then H is also a weak bialgebra whose algebra and coalgebra structures are given by direct sums. The target and source counital maps ε_t and ε_s are given by

$$\varepsilon_t(x) = (\varepsilon_A)_t(a) + (\varepsilon_B)_t(b),$$
$$\varepsilon_s(x) = (\varepsilon_A)_s(a) + (\varepsilon_B)_s(b),$$

for $x = a + b \in H \ (a \in A,\ b \in B)$. Here, $(\varepsilon_A)_t, (\varepsilon_A)_s$ are the target and source counital maps of A, and $(\varepsilon_B)_t, (\varepsilon_B)_s$ are that of B.

If A, B are WHAs with antipodes S_A, S_B, then H is also a WHA with antipode S given by

$$S(a + b) = S_A(a) + S_B(b) \qquad (a \in A,\ b \in B).$$

A weak bialgebra (resp. WHA) H is called indecomposable if there are no weak bialgebras (resp. WHA) A, B such that $H \cong A \oplus B$.

Theorem 8 (Decomposition theorem)

Let H be a finite-dimensional weak bialgebra. Then

(1) there are finitely many indecomposable weak bialgebras H_i $(i = 1, \ldots, k)$ such that $H = H_1 \oplus \cdots \oplus H_k$.

(2) Let H_i $(i = 1, \ldots, k)$ and H'_j $(j = 1, \ldots, l)$ be indecomposable weak bialgebras satisfying $H_1 \oplus \cdots \oplus H_k = H = H'_1 \oplus \cdots \oplus H'_l$. Then $k = l$, and $H'_j = H_{\sigma(j)}$ $(j = 1, \ldots, l)$ for some permutation $\sigma \in \mathfrak{S}_l$.

This result follows from existence and uniqueness of decompositions into direct sums of indecomposable ideals for finite-dimensional algebras.

Let A, B be two finite-dimensional WHAs, and consider the direct sum $H := A \oplus B$. Let $\pi_A : H \longrightarrow A$, $\pi_B : H \longrightarrow B$ be the natural projections. Then A^* and B^* can be regarded as subcoalgebras of H^* via the transposed maps ${}^t\pi_A : A^* \longrightarrow H^*$, ${}^t\pi_B : B^* \longrightarrow H^*$. Moreover,

Lemma 9

The dual WHA H^* is isomorphic to the direct sum of the dual WHAs A^* and B^*: $H^* = A^* \oplus B^*$.

By this lemma we have:

Proposition 10

A finite-dimensional weak bialgebra H is indecomposable as a weak bialgebra if and only if H^* is so.

Theorem 11

A finite-dimensional bialgebra H is indecomposable as a WBA.

(**Proof**)

Suppose that $H = A \oplus B$ for some WBAs A and B. Then $\mathrm{End}_H(H_t) \cong \mathrm{End}_A(A_t) \oplus \mathrm{End}_B(B_t)$ as vector spaces. Thus,

$$\begin{aligned} \dim \mathrm{End}_H(H_t) &= \dim \mathrm{End}_A(A_t) + \dim \mathrm{End}_B(B_t) \\ &\geq 1 + 1 \geq 2. \end{aligned}$$

This contradicts what $\dim \mathrm{End}_H(H_t) = 1$ since $H_t = \boldsymbol{k}1_H$. □

Example 12

For any finite group G, the group Hopf algebra $\boldsymbol{k}[G]$ and its dual Hopf algebra $(\boldsymbol{k}[G])^*$ are indecomposable weak bialgebras.

Problem 13

(1)† Is there a finite-dimensional indecomposable WHA such that it is not a Hopf algebra?

(2) For any finite-dimensional weak bialgebra over $\boldsymbol{k}$, can $\varepsilon(1)$ be written as $n1_{\boldsymbol{k}}$ for some positive integer n?

Remark. Problem (1) replaced by "bialgebra" instead of "Hopf algebra" is affirmative solved.

Let us examine some properties of preserving under the direct sum construction.

Definition 14 ([1])

Let H be a weak bialgebra over $\boldsymbol{k}$.

(1) $\Lambda \in H$ is a **left integral** if $x\Lambda = \varepsilon_t(x)\Lambda$ for all $x \in H$.

(2) $\Lambda \in H$ is a **right integral** if $\Lambda x = \Lambda\varepsilon_s(x)$ for all $x \in H$.

† After my presentation, from several experts I received several ideas for solving this problem. I would like to express gratitude for all.

Definition 14 (continued)

(3) $$G(H) = \left\{ g \in H \;\middle|\; \begin{array}{l} \Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g), \\ g \text{ is invertible} \end{array} \right\}.$$

An element in $G(H)$ is called a **group-like element**.

Remark 15

1. $G(H)$ becomes a group with respect to the product in H.
2. If H has an antipode, then for any $g \in H$ satisfying $\Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g)$,

$$\varepsilon_s(g) = \varepsilon_t(g) = 1 \iff g \text{ is invertible in } H.$$

The concepts of quasitriangular and ribbon structures for WHAs were introduced by Nikshych, Turaev and Vainerman [6].

[6] D. Nikshych, V. Turaev and L. Vainerman, "Invariants of knots and 3-manifolds from finite quantum groupoids", Top. Appl. 127 (2003), 91–123.

Proposition 16

Let A, B be two finite-dimensional WHAs, and $H = A \oplus B$ be the direct sum of them. Then,

(1) H is (co)semisimple if and only if A, B are (co)semisimple,

(2) between the sets of left integrals $\mathscr{I}^L(A), \mathscr{I}^L(B), \mathscr{I}^L(H)$,

$$\mathscr{I}^L(H) = \{ \ \Lambda_A + \Lambda_B \mid \Lambda_A \in \mathscr{I}^L(A), \ \Lambda_B \in \mathscr{I}^L(B) \ \},$$

(3) as groups

$$G(H) \cong G(A) \times G(B),$$

(4) any universal R-matrix of H is expressed as $R = R_A + R_B$ where R_A, R_B are universal R-matrices of A, B, respectively. Conversely, for universal R-matrices R_A, R_B of A, B, respectively, $R := R_A + R_B$ is a universal R-matrix of H.

Example 17

Let us consider two Taft algebras $H_{m^2}(\omega)$ and $H_{n^2}(\lambda)$, where ω and λ are primitive mth and nth roots of unity in $\boldsymbol{k}$, respectively. Then, we have the direct sum
$H := H_{m^2}(\omega) \oplus H_{n^2}(\lambda)$.
In particular, we consider the case where $m = n = 2$, and $\omega = \lambda = -1$. $H_4(-1)$ is called Sweedler's 4-dimensional Hopf algebra, and $\dim H = 8$. As an algebra,

$$H = \left\langle e_1, e_2, g, h, x, y \;\middle|\; \begin{array}{l} g^2 = e_1,\ h^2 = e_2,\ x^2 = y^2 = 0, \\ xg = -gx,\ yh = -hy, \\ e_1 + e_2 = 1,\ ab = ba = 0, \\ ae_1 = e_1 a = a,\ be_2 = e_2 b = b \\ (a \in \{e_1, g, x\},\ b \in \{e_2, h, y\}) \end{array} \right\rangle.$$

By Radford, it is shown that if the characteristic of $\boldsymbol{k}$ is not 2, then the universal R-matrices of $H_4(-1)$ are parametrized by $\alpha \in \boldsymbol{k}$, and they are given by

Example 17 (continued)

$$R_\alpha = \frac{1}{2}(e \otimes e + g \otimes e + e \otimes g - g \otimes g) + \frac{\alpha}{2}(x \otimes x + x \otimes gx + gx \otimes gx - gx \otimes x).$$

Therefore, the universal R-matrices of H are parametrized by $\alpha, \beta \in \boldsymbol{k}$, and are given by

$$\begin{aligned} R_\alpha + R_\beta = {} & \frac{1}{2}(e_1 \otimes e_1 + g \otimes e_1 + e_1 \otimes g - g \otimes g) \\ & + \frac{\alpha}{2}(x \otimes x + x \otimes gx + gx \otimes gx - gx \otimes x) \\ & + \frac{1}{2}(e_2 \otimes e_2 + h \otimes e_2 + e_2 \otimes h - h \otimes h) \\ & + \frac{\beta}{2}(y \otimes y + y \otimes hy + hy \otimes hy - hy \otimes y). \end{aligned}$$

For two qtWHAs (A, R_A), (B, R_B), we define a qtWHA by the direct sum

$$(A, R_A) \oplus (B, R_B) := (A \oplus B, R_A + R_B).$$

Theorem 18

Let A, B be two finite-dimensional WHAs, and consider the direct sum $H := A \oplus B$. Then, the quantum double $D(H)$ is isomorphic to the direct sum of the quantum doubles $D(A)$ and $D(B)$: $D(H) = D(A) \oplus D(B)$.

Proposition 19

Let (A, R_A), (B, R_B) be two qtWHAs of finite dimension, and (H, R) be their direct sum. Then the map

$$\begin{array}{ccc} \mathrm{Rib}(A, R_A) \times \mathrm{Rib}(B, R_B) & \longrightarrow & \mathrm{Rib}(H, R) \\ (v_A, v_B) & \longmapsto & v_A + v_B \end{array}$$

is bijective, where $\mathrm{Rib}(H, R)$ is the set of ribbon elements of (H, R).

§3. A Kaplansky type construction for WBAs

Due to Chebel and Makhlouf [3], we call the following construction a Kaplansky type construction for WBAs.

Theorem 20 (Chebel and Makhlouf)

Let $A = (A, \Delta_A, \varepsilon_A)$ be a bialgebra over $\boldsymbol{k}$, and introduce a new element $1 \notin A$. As a vector space we set $H := A \oplus \boldsymbol{k}1$, and extend the multiplication in A to that in H as follows:

$$1 \cdot a = a = a \cdot 1, \quad 1 \cdot 1 = 1 \quad (a \in A).$$

Furthermore, define two $\boldsymbol{k}$-linear maps $\Delta : H \longrightarrow H \otimes H$, $\varepsilon : H \longrightarrow \boldsymbol{k}$ by for all $a \in A$

$$\Delta(a) = \Delta_A(a), \qquad \varepsilon(a) = \varepsilon_A(a),$$

$$\Delta(1) = (1 - e) \otimes (1 - e) + e \otimes e, \qquad \varepsilon(1) = 2.$$

Then H is a weak bialgebra. If A is a Hopf algebra with antipode S_A, then H becomes a WHA with antipode S, which is defined by $S(a) = S_A(a)$ $(a \in A)$ and $S(1) = 1$.

Example 21 (Taft's weak Hopf algebra [3])

Let $n \geq 2$ be an integer, and $\boldsymbol{k}$ be a field which contains a primitive nth root of unity $\lambda \in \boldsymbol{k}$. Let $H_{n^2}(\lambda)$ be the n^2-dimensional Taft algebra, that is,

$$H_{n^2}(\lambda) = \langle\ g, x \mid g^n = e,\ x^n = 0,\ xg = \lambda gx\ \rangle,$$

where e is the identity element. Applying Theorem 20 we have (n^2+1)-dimensional weak Hopf algebra $H'_{n^2}(\lambda)$. Its structure maps are given as follows with identity element 1:

$$\begin{aligned}
&\Delta(1) = (1-e)\otimes(1-e) + e\otimes e, && \Delta(e) = e\otimes e,\\
&\Delta(g) = g\otimes g, && \Delta(x) = g\otimes x + x\otimes e,\\
&\varepsilon(1) = 2, && \varepsilon(e) = 1,\\
&\varepsilon(g) = 1, && \varepsilon(x) = 0,\\
&S(1) = 1, && S(e) = e,\\
&S(g) = g^{-1}, && S(x) = -g^{-1}x.
\end{aligned}$$

The Kaplansky type construction in Theorem 20 can be regarded as a special direct sum construction for weak bialgebras.

Theorem 22

Let A be a bialgebra over $\boldsymbol{k}$ with identity element e, and $H = A \oplus \boldsymbol{k}1$ be the weak bialgebra obtained by the Kaplansky type construction from A. Then, $\boldsymbol{k}(1-e)$ is a two-sided ideal and a subcoalgebra of H, and $H = A \oplus \boldsymbol{k}(1-e)$ as weak bialgebras.

(**Proof**)

This can be verified by direct computation. □

§4. Structures of 2 and 3-dimensional WBAs

Chebel and Makhlouf [3] classified two and three dimensional weak bialgebras over $\mathbb{C}$ up to isomorphism.

Proposition 23 (Chebel and Makhlouf [3; Prop. 4.3])

In the 2-dimensional weak bialgebras over $\mathbb{C}$, there are exactly three isomorphism classes, and their representatives are given by $H = \mathbb{C}e_1 + \mathbb{C}e_2$ with multiplication m, comultiplication Δ and counit ε defined below:

$$m(e_1, e_1) = e_1,\ m(e_1, e_2) = m(e_2, e_1) = m(e_2, e_2) = e_2,$$

(#1) $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = e_2 \otimes e_2,$
$\varepsilon(e_1) = \varepsilon(e_2) = 1.$

(#2) $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2,$
$\varepsilon(e_1) = \varepsilon(e_2) = 1.$

(#3) $\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2,\ \Delta(e_2) = e_2 \otimes e_2,$
$\varepsilon(e_1) = 2,\ \varepsilon(e_2) = 1.$

(#1) $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = e_2 \otimes e_2,$
$\varepsilon(e_1) = \varepsilon(e_2) = 1.$

(#2) $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2,$
$\varepsilon(e_1) = \varepsilon(e_2) = 1.$

(#3) $\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2,\ \Delta(e_2) = e_2 \otimes e_2,$
$\varepsilon(e_1) = 2,\ \varepsilon(e_2) = 1.$

Remark.

1. The weak bialgebras (#2) and (#3) are WHAs since one can find antipodes S defined by $S(e_1) = e_1,\ S(e_2) = e_2$ [3; Proposition 4.4]. The weak bialgebra (#3) is one and only such that it is not a bialgebra.
2. The weak bialgebra (#2) is isomorphic to the group Hopf algebra $\mathbb{C}[G]$ of $G = \mathbb{Z}/2\mathbb{Z}$.
3. The weak bialgebras (#1) and (#2) are indecomposable. On the other hand, (#3) can be decomposed as $\mathbb{C}(e_1 - e_2) \oplus \mathbb{C}e_2 \cong \mathbb{C} \oplus \mathbb{C}$ as a weak bialgebra.

Proposition 24 (Chebel and Makhlouf [3; Prop. 4.5])

In the 3-dimensional weak bialgebras over $\mathbb{C}$, there are exactly 20 isomorphism classes (#1),. . .,(#20). The isomorphism types of them as algebras are the following*:

$$\mathbb{C} \times \mathbb{C} \times \mathbb{C},\ \mathbb{C}[t]/(t^2) \times \mathbb{C},\ \mathrm{T}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle|\ a, b, c \in \mathbb{C} \right\}.$$

(1) On $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ there are exactly 11 WBA structures.

(2) On $\mathbb{C}[t]/(t^2) \times \mathbb{C}$ there are exactly 4 WBA structures.

(3) On $\mathrm{T}_2(\mathbb{C})$ there are exactly 5 WBA structures.

Among them, the number of WHAs is 3, and all such WHAs are contained in the class (1). The number of WBAs which are not bialgebras is 5.

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*thanks to helpful comments from Noriyuki Suwa at H-ACT 2019, Tsukuba University

Remark 25

1. Among the 3-dimensional WBAs except for (#8), (#9), (#10) are indecomposable as weak bialgebras.
   The WBAs (#8), (#9) and (#10) can be decomposed into direct sums of indecomposable weak bialgebras as follows:
   $(\#8) = \mathbb{C} \oplus (\text{Prop.23}(\#1))^{\dagger}, \quad (\#9) = \mathbb{C} \oplus \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^{\dagger},$
   $(\#10) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$
2. the weak bialgebra (#1), that is a group Hopf algebra of $G = \mathbb{Z}/3\mathbb{Z}$, is a unique 3-dimensional WHA which is indecomposable.
3. The weak bialgebras (#1), (#2), (#5), (#8), (#9), (#10), (#15) are self dual, and
   $$(\#3)^* = (\#7), \quad (\#4)^* = (\#13), \quad (\#5)^* = (\#20),$$
   $$(\#6)^* = (\#18), \quad (\#11)^* = (\#16), \quad (\#12)^* = (\#19),$$
   $$(\#14)^* = (\#17).$$

† **corrected on January 16, 2020.**

List of $G(H)$, $\mathscr{I}^l(H)$, $\mathscr{I}^r(H)$ for WBAs H of dim ≤ 3.

In the following list e_1 stands for the identity element.

H	$G(H)$	$\mathscr{I}^l(H)$	$\mathscr{I}^r(H)$
Prop.23 #1	$\{e_1\}$	$\mathbb{C}e_2$	$\mathbb{C}e_2$
Prop.23 #2	$\{e_1, -e_1+2e_2\}$	$\mathbb{C}e_2$	$\mathbb{C}e_2$
Prop.23 #3	$\{e_1\}$	H	H
Prop.24 #1	$(*1)$	$\mathbb{C}e_3$	$\mathbb{C}e_3$
Prop.24 #2	$\{e_1\}$	$\mathbb{C}e_3$	$\mathbb{C}e_3$
Prop.24 #3	$\{e_1\}$	$\mathbb{C}(e_2-e_3)$	$\mathbb{C}(e_2-e_3)$
Prop.24 #4	$\{e_1\}$	$\mathbb{C}e_3$	$\mathbb{C}e_3$
Prop.24 #5	$\{e_1\}$	$\mathbb{C}(e_2-e_3)$	$\mathbb{C}(e_2-e_3)$
Prop.24 #6	$\{e_1\}$	$\mathbb{C}e_3$	$\mathbb{C}e_3$
Prop.24 #7	$\{e_1, -e_1+2e_2\}$	$\mathbb{C}e_3$	$\mathbb{C}e_3$

where $(*1) = \{\ e_1,\ \omega e_1 - (1+2\omega)e_2 + (2+\omega)e_3,$ $\omega^2 e_1 + (1+2\omega)e_2 + (1-\omega)e_3\ \} \cong \mathbb{Z}/3\mathbb{Z}$, and ω is a primitive 3rd of unity.

H	$G(H)$	$\mathscr{I}^l(H)$	$\mathscr{I}^r(H)$
Prop.24 #8	$\{e_1\}$	$(*2)$	$(*2)$
Prop.24 #9	$\{e_1, e_1 - 2e_3\}$	$(*3)$	$(*3)$
Prop.24 #10	$\{e_1\}$	H	H
Prop.24 #11	$(*4)$	$\mathbb{C}e_2 + \mathbb{C}e_3$	$\mathbb{C}e_2 + \mathbb{C}e_3$
Prop.24 #12	$\{e_1\}$	$\mathbb{C}(e_1 - e_2)$	$\mathbb{C}(e_1 - e_2)$
Prop.24 #13	$\{e_1\}$	$\mathbb{C}(e_1 - e_2)$	$\mathbb{C}(e_1 - e_2)$
Prop.24 #14	$\{e_1\}$	$\mathbb{C}(e_1 - e_2)$	$\mathbb{C}(e_1 - e_2)$
Prop.24 #15	$\{e_1\}$	$\mathbb{C}(e_1 - e_2)$	$\mathbb{C}(e_1 - e_2)$

where $(*2) = \mathbb{C}(e_1 - e_2) + \mathbb{C}e_3$,
$(*3) = \mathbb{C}(e_1 - e_2) + \mathbb{C}(e_1 - e_3)$,
$(*4) = \{\ ae_1 + (1-a)e_2 \mid a \in \mathbb{C} - \{0\}\ \}$.
In $(*4)$ since
$(ae_1 + (1-a)e_2)(be_1 + (1-b)e_2) = abe_1 + (1-ab)e_2$
for $a, b \in \mathbb{C} - \{0\}$, $G(H)$ is isomorphic to the multiplicative group of $\mathbb{C} - \{0\}$.

H	$G(H)$	$\mathscr{I}^l(H)$	$\mathscr{I}^r(H)$
Prop.24 #16	$\{e_1\}$	$\mathbb{C}(e_1 - e_2 + e_3)$	$\mathbb{C}(e_2 + e_3)$
Prop.24 #17	$\{e_1\}$	$\{0\}$	$(*5)$
Prop.24 #18	$\{e_1\}$	$\{0\}$	$(*5)$
Prop.24 #19	$\{e_1\}$	$\mathbb{C}e_2 + \mathbb{C}e_3$	$\{0\}^\dagger$
Prop.24 #20	$\{e_1\}$	$\mathbb{C}e_2 + \mathbb{C}e_3$	$\{0\}^\dagger$

where $(*5) = \mathbb{C}(e_1 - e_2) + \mathbb{C}e_3$.

Proposition 26 (QT structures of low dim. WHAs, Zhang, Zhao and Wang [13])

(1) The 2-dimensional WHA (#3) and the 3-dimensional WHA (#10) have a unique universal R-matrix, which is given by $\Delta(e_1)$.

(2) The 3-dimensional WHA (#9) has exactly two universal R-matrices, which are given by $\Delta(e_1),\ \Delta(e_1) - 2\,e_3 \otimes e_3$.

[13] X. Zhang, X. Zhao and S. Wang, "Sovereign and ribbon weak Hopf algebras", Kodai Math. J. 38 (2015), 451–469.

† corrected after my presentation

(**Proof**)

It follows from Proposition 16(4) and Remark 25.1. □

Since $\mathbb{C}[G]$ of the cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ has exactly m universal R-matrices, we see that:

Corollary 27

(1) Isomorphism classes of the 2-dimensional WHAs over $\mathbb{C}$ are determined by the number of universal R-matrices.

(2) The same statement hold for the 3-dimensional WHAs over $\mathbb{C}$.

Proposition 28 (Structures of the duals and the quantum doubles of 3-dimensional WHAs)

(1) In the case of $H = (\#9)$, H^* is isomorphic to H, and $D(H)$ is a 5-dimensional WHA that is commutative and cocommutative. In particular, it is not isomorphic to the 5-dimensional Taft's weak algebra.

Proposition 28 (Structures of the duals and the quantum doubles of 3-dimensional WHAs (continued))

(2) In the case of $H = (\#10)$, both of H^* and $D(H)$ are isomorphic to H.

(Proof)

$$\begin{aligned}
\textbf{(1)} \quad (\#9)^* &\cong \mathbb{C}^* \oplus (\textbf{Prop.23}(\#2))^* \\
&\cong \mathbb{C} \oplus (\textbf{Prop.23}(\#2)) = (\#9), \\
D(\#9) &\cong D(\mathbb{C}) \oplus D(\textbf{Prop.23}(\#2)) \\
&\cong D(\mathbb{C}) \oplus D(\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]) = \mathbb{C} \oplus \mathbb{C}[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]
\end{aligned}$$

(2) It follows from $(\#10) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. □

Remark. Part (1) has shown by Zhang, Zhao and Wang [13]. In their paper, the ribbon elements of $(D(H), \mathcal{R})$ are determined. This result can be confirmed by Proposition 19.

Unit objects in module categories over WBAs

The module category ${}_H\mathsf{M}$ over a WBA H has a structure of k-linear monoidal category [5]. The tensor product of two left H-modules V and W are defined by

$$V \circledast W := \Delta(1) \cdot (V \otimes W),$$

where $\cdot$ indicates the diagonal action on $V \otimes W$.
The unit object in ${}_H\mathsf{M}$ is the target subalgebra H_t together with the action

$$x \cdot z = \varepsilon_t(xz) \qquad (x \in H,\ z \in H_t). \tag{4}$$

This module is called the trivial left H-module.

Structures of the trivial module

Lemma 29

Let H be a WBA over $\boldsymbol{k}$. Then, $(H_s)^* \cong H_t$ as left H-modules, where the left H-action on $(H_s)^*$ is given by

$$(x \cdot p)(y) := p\big(\varepsilon_s(yx)\big) \quad (x \in H,\ p \in (H_s)^*,\ y \in H_s).$$

Let $Z(H)$ denote the center of H, and set

$$\boldsymbol{Z_t := H_t \cap Z(H), \qquad Z_s := H_s \cap Z(H).}$$

Proposition 30 (Böhm, Nill and Szlachányi [1; Prop. 2.15])

Let H be a WBA over $\boldsymbol{k}$. Denoted by $D_\varepsilon : H \longrightarrow \mathrm{End}(V_\varepsilon)$ is the representation corresponding to the action of $V_\varepsilon := (H_s)^*$ given in Lemma 29. Then

$$\mathrm{End}_H(V_\varepsilon) = D_\varepsilon(Z_t) = D_\varepsilon(Z_s).$$

Remark. From the above proposition the indecomposable components of the trivial H-module are multiplicity free [1].

List of decompositions of the trivial H-modules into indecomposable components for H of dim ≤ 3.

By computing the primitive idempotents of the algebra $\mathrm{End}_H(H_t)$ thanks to Proposition 30, we have the following table:

H	decomp. of H_t into indec. comps.
Prop.23 #1	$\mathbb{C}e_1$
Prop.23 #2	$\mathbb{C}e_1$
Prop.23 #3	$H = \mathbb{C}e_2 \oplus \mathbb{C}(e_1 - e_2)$
Prop.24 #1	$\mathbb{C}e_1$
Prop.24 #2	$\mathbb{C}e_1$
Prop.24 #3	$\mathbb{C}e_1$
Prop.24 #4	$\mathbb{C}e_1$
Prop.24 #5	$\mathbb{C}e_1$
Prop.24 #6	$\mathbb{C}e_1$
Prop.24 #7	$\mathbb{C}e_1$
Prop.24 #8	$\mathbb{C}(e_1 - e_2) \oplus \mathbb{C}e_2$

H	decomp. of H_t into indec. comps.
Prop.24 #9	$\mathbb{C}(e_1 - e_2) \oplus \mathbb{C}e_2$
Prop.24 #10	$H = \mathbb{C}(e_1 - e_2) \oplus \mathbb{C}(e_2 - e_3) \oplus \mathbb{C}e_3$
Prop.24 #11	$\mathbb{C}(e_1 - e_2 + e_3) \oplus \mathbb{C}(e_2 - e_3)$
Prop.24 #12	$\mathbb{C}e_1$
Prop.24 #13	$\mathbb{C}e_1$
Prop.24 #14	$\mathbb{C}e_1$
Prop.24 #15	$\mathbb{C}e_1$
Prop.24 #16	$\mathbb{C}(e_1 - e_2) \oplus \mathbb{C}e_2$
Prop.24 #17	$\mathbb{C}e_1$
Prop.24 #18	$\mathbb{C}e_1$
Prop.24 #19	$\mathbb{C}e_1$
Prop.24 #20	$\mathbb{C}e_1$

Among the 2 and 3-dimensional weak bialgebras H, the trivial H-module is decomposable if and only if H is not a bialgebra. So, we state the following problem:

Problem 31

Is it true that the trivial H-module is decomposable for a weak bialgebra H that is not a bialgebra?

Remark 32

Since the WBA (#16) is indeconposable as an algebra, it is also indecompposable as a weak bialgbra. Nevertheless, it is remarkable that the trivial module is decomposable.

§5. A categorical interpretation of indecomposability

Notation. **For a WBA H,**

$_H\mathsf{M}$:= (the monoidal category of left H-modules and H-linear maps),

$_H\mathbb{M}$:= (the full subcategory of $_H\mathsf{M}$ whose objects are finite-dimensional).

Lemma 33

Let A and B be two WBAs over $\boldsymbol{k}$, and consider the direct sum WBA $H = A \oplus B$. Then, any left H-module X is decomposed as $X = (1_A \cdot X) \oplus (1_B \cdot X)$. This decomposition gives rise to identical equivalences $_H\mathsf{M} \simeq {}_A\mathsf{M} \times {}_B\mathsf{M}$ and $_H\mathbb{M} \simeq {}_A\mathbb{M} \times {}_B\mathbb{M}$ as $\boldsymbol{k}$-linear monoidal categories.

A $\boldsymbol{k}$-linear monoidal category $\mathscr{C}$ is called indecomposable if $\mathscr{C}$ can not be decomposed to a direct sum $\mathscr{C}_1 \times \mathscr{C}_2$ for some $\boldsymbol{k}$-linear monoidal categories $\mathscr{C}_1, \mathscr{C}_2$. If not, then $\mathscr{C}$ is called decomposable.

By Lemma 33 we have:

Corollary 34

Let H be a decomposable WBA over $\boldsymbol{k}$. Then the $\boldsymbol{k}$-linear monoidal categories ${}_H\mathsf{M}$ and ${}_H\mathbb{M}$ are decomposable.

The "converse" is true.

Theorem 35 (A categorical characterization of indecomposable WBAs)

Let H be a finite-dimensional WBA over $\boldsymbol{k}$. Then, H is indecomposable as a WBA if and only if the $\boldsymbol{k}$-linear monoidal category ${}_H\mathbb{M}$ is indecomposable.

Notation. **For a coalgebra C,**

M^C **:= (the k-linear abelian category of right C-comodules and C-colinear maps).**

Let H be a WBA over k. Any right H-comodule V has an (H_s, H_s)-bimodule structure defined as follows: for $y \in H_s$ and $v \in V$,

$$y \cdot v = v_{(0)}\varepsilon(yv_{(1)}), \tag{5}$$

$$v \cdot y = v_{(0)}\varepsilon(v_{(1)}y). \tag{6}$$

V can be regarded as a right H-comodule in the monoidal category ${}_{H_s}\mathsf{M}_{H_s}$ since the coaction of V is (H_s, H_s)-linear map.
Consider the subcategory ${}_{H_s}\mathsf{M}^H_{H_s}$ of ${}_{H_s}\mathsf{M}_{H_s}$, whose objects are right H-comodules and morphisms are (H_s, H_s)-linear maps preserving H-comodule structures. Then, we have an equivalence

$$\Xi : \mathsf{M}^H \longrightarrow {}_{H_s}\mathsf{M}^H_{H_s}$$

of k-linear abelian categories since a right H-comodule map $f : M \longrightarrow N$ is always (H_s, H_s)-linear map.

We have the composition

$$\hat{U}^H : \mathsf{M}^H \xrightarrow{\ \Xi\ } {}_{H_s}\mathsf{M}^H_{H_s} \xrightarrow{\text{forgetful}} {}_{H_s}\mathsf{M}_{H_s}.$$

$\hat{U}^H$ is also said to be a forgetful functor.

Lemma 36 ([7; Lemma 4.2])

Let H be a WBA over $\boldsymbol{k}$. Then M^H has a structure of $\boldsymbol{k}$-linear monoidal category such that $\hat{U}^H$ is a $\boldsymbol{k}$-linear monoidal functor. Moreover, the equivalence $\Xi : \mathsf{M}^H \longrightarrow {}_{H_s}\mathsf{M}^H_{H_s}$ becomes an equivalence of $\boldsymbol{k}$-linear monoidal category.

Remark. Lemma 36 is extended to a more general setting by Szlachányi [10; Theorem 2.2].

[7] F. Nill, "Axioms for weak bialgebras", arXiv:math.9805104v1, 1998.

[10] K. Szlachányi, "Adjointable monoidal functors and quantum groupoids", In: "Hopf algebras in noncommutative geometry and physics", Lecture Notes in Pure and Appl. Math. 239, 291–307, Dekker, New York, 2005.

Let $(\mathscr{C}, \otimes, I)$, $(\mathscr{D}, \otimes', I')$ be two monoidal categories. A triad $(F, \bar{\phi}^F, \bar{\omega}^F)$ consisting of

- a covariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$,
- a natural transformation
$$\bar{\phi}^F = \{\bar{\phi}^F_{X,Y} : F(X \otimes Y) \longrightarrow F(X) \otimes' F(Y)\}_{X,Y \in \mathscr{C}},$$
- a morphism $\bar{\omega}^F : F(I) \longrightarrow I'$

is said to be comonoidal if they satisfy some compatibility conditions [2; Subsections 1.5–1.6].
A comonoidal functor $(F, \bar{\phi}^F, \bar{\omega}^F)$ is called strong if $\bar{\phi}^F$ is a natural equivalence and $\bar{\omega}^F$ is an isomorphism. A strong comonoidal functor can be regarded as a strong monoidal functor.

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[2] A. Bruguieres and A. Virelizier, "Hopf monads", Adv. Math. 215 (2007), 679–733.

Lemma 37

Let H, K be two WBAs over $\boldsymbol{k}$, and $\varphi : H \longrightarrow K$ be a weak bialgebra map. Then,

(1) For a right H-comodule (M, ρ_M)

$$\mathbf{M}^{\varphi}(M, \rho_M) := (M, (\mathrm{id}_M \otimes \varphi) \circ \rho_M)$$

is a right K-comodule, and for a right H-comodule map $f : (M, \rho_M) \longrightarrow (N, \rho_N)$

$$\mathbf{M}^{\varphi}(f) := f : \mathbf{M}^{\varphi}(M, \rho_M) \longrightarrow \mathbf{M}^{\varphi}(N, \rho_N)$$

is a right K-comodule map. In this way, a covariant functor $\mathbf{M}^{\varphi} : \mathbf{M}^{H} \longrightarrow \mathbf{M}^{K}$ is obtained.

(2) The functor $\mathbf{M}^{\varphi}$ becomes a $\boldsymbol{k}$-linear comonoidal. If $\varphi_s := \varphi|_{H_s} : H_s \longrightarrow K_s$ is bijective, then $\mathbf{M}^{\varphi}$ is strong.

(3) The algebra map φ_s induces a $\boldsymbol{k}$-linear monoidal functor ${}_{\varphi_s}\mathbf{M}_{\varphi_s} : {}_{K_s}\mathbf{M}_{K_s} \longrightarrow {}_{H_s}\mathbf{M}_{H_s}$, and if φ_s is bijective, then $\hat{U}^K \circ \mathbf{M}^{\varphi} = {}_{\varphi_s^{-1}}\mathbf{M}_{\varphi_s^{-1}} \circ \hat{U}^H$ as monoidal functors.

Notation.

$$\mathbf{Vect}_{\boldsymbol{k}}^{\mathbf{f.d.}} = (\text{the } k\text{-linear category consisting of finite-dimensional vector spaces and } k\text{-linear maps between them})$$

For a coalgebra C

$$\mathbb{M}^C = (\text{the full subcategory } \mathsf{M}^C\text{, whose objects are finite-dimensional right } C\text{-comodules}),$$

and $U^C : \mathbb{M}^C \longrightarrow \mathbf{Vect}_{\boldsymbol{k}}^{\mathbf{f.d.}}$ denotes the forgetful functor.

The following theorem is fundamental on Tannakian reconstruction theory.

Theorem 38 (Reconstruction of a coalgebra map)

Let C, D be two coalgebras over $\boldsymbol{k}$, and $F : \mathbb{M}^C \longrightarrow \mathbb{M}^D$ be a $\boldsymbol{k}$-linear functor. If $U^D \circ F = U^C$, then there is a unique coalgebra map $\varphi : C \longrightarrow D$ such that $F = \mathbb{M}^\varphi$, where $\mathbb{M}^\varphi$ is the $\boldsymbol{k}$-linear functor induced from φ.

(**Proof** referred from Franco [4])

Let (M, ρ_M) be a finite-dimensional right C-comodule. Since $U^D \circ F = U^C$, we have $F(M, \rho_M) = (M, \rho_M^F)$. Let P be a finite-dimensional subcoalgebra of C and we regard it a right C-comodule by

$$\rho_P : P \xrightarrow{\Delta_P} P \otimes P \xrightarrow{\mathrm{id} \otimes \iota_P} P \otimes C,$$

where ι_P is an inclusion. Then we have $F(P, \rho_P) = (P, \rho_P^F) \in \mathbb{M}^D$. Consider the composition

$$\varphi_P : P \xrightarrow{\rho_P^F} P \otimes D \xrightarrow{\varepsilon_P \otimes \mathrm{id}} k \otimes D \cong D.$$

We see that $\varphi_P : P \longrightarrow D$ is a coalgebra map. By the fundamental theorem of coalgebras, C is a sum of finite-dimensional subcoalgebras. From this fact, we obtain a coalgebra map $\varphi : C \longrightarrow D$ by pasting all φ_P. It can be shown that φ satisfies a unique coalgebra map such that $F = \mathbb{M}^\varphi$. □

[4] I.L. Franco, "Topics in category theory: Hopf algebras", a lecture note, noted by D. Mehrle, at Cambridge University, 2015.

Theorem 39 (Reconstruction of a WBA map)

Let A, B be two WBAs over $\boldsymbol{k}$, and $F : \mathbb{M}^A \longrightarrow \mathbb{M}^B$ be a strong $\boldsymbol{k}$-linear comonoidal functor. If $U^B \circ F = U^A$ as $\boldsymbol{k}$-linear monoidal functors, then there is a unique WBA map $\varphi : A \longrightarrow B$ such that $F = \mathbb{M}^\varphi$ as $\boldsymbol{k}$-linear comonoidal functors, and $\bar{\omega}^F = \varphi|_{A_s} : A_s \longrightarrow B_s$ is an isomorphism of algebras. Furthermore, the equation $\hat{U}^B \circ F = {}_{\varphi_s^{-1}}\mathbf{M}_{\varphi_s^{-1}} \circ \hat{U}^A$ holds.

(**Proof**)

By Theorem 38 there is a unique coalgebra map $\varphi : A \longrightarrow B$ such that $F = \mathbb{M}^\varphi$ as k-linear functors. Since $U^B \circ F = U^A$ as k-linear monoidal functors, we see that

$$\bar{\phi}^F_{M,N} : F(M \otimes_{A_s} N) \longrightarrow F(M) \otimes_{B_s} F(N)$$

is induced from $\mathrm{id}_{M \otimes N}$ for all $M, N \in \mathbb{M}^A$.

It can be shown that

(1) **φ is an algebra map,**
(2) **$\bar{\omega}^F = \varphi|_{A_s} : A_s \longrightarrow B_s$ is an isomorphism of algebras,**
(3) **$F = \mathbb{M}^{\varphi}$ as k-linear comonoidal functors.**

Finally, by Lemma 37, $\hat{U}^B \circ F = {}_{\varphi_s^{-1}}\mathsf{M}_{\varphi_s^{-1}} \circ \hat{U}^A$. □

The following is a classical result known as a bialgebra version of Tannakian reconstruction theorem.

Theorem 40 (Ulbrich[12], Schauenburg[8; Theorem 5.4])

Let $\mathscr{C}$ be a $\boldsymbol{k}$-linear monoidal category, and $\omega : \mathscr{C} \longrightarrow \mathrm{Vect}_{\boldsymbol{k}}^{\mathrm{f.d.}}$ be a faithful and exact $\boldsymbol{k}$-linear monoidal functor. Then there are a bialgebra B and a monoidal category equivalence $F : \mathscr{C} \longrightarrow \mathbb{M}^B$ such that $U^B \circ F = \omega$.

[8] P. Schauenburg, "Hopf bigalois extensions", Comm. Algebra 24 (1996), 3797–3825.

[12] K.-H. Ulbrich, "On Hopf algebras and rigid monoidal categories", Israel J. Math. 72 (1990), 252–256.

By using Theorems 39 and 40 one can show Theorem 35 (A categorical characterization of indecomposable WBAs).

(Proof of Theorem 35)
"Only if" part follows from Corollary 34.
"If" part can be shown as follows. Assume that H is indecomposable as a WBA, but ${}_H\mathbb{M}$ is not. Then there are two k-linear monoidal categories $\mathscr{C}_1, \mathscr{C}_2$ such that ${}_H\mathbb{M} \simeq \mathscr{C}_1 \times \mathscr{C}_2$. Let $F : \mathscr{C}_1 \times \mathscr{C}_2 \longrightarrow {}_H\mathbb{M}$ be a k-linear monoidal category equivalence. Since k-linear monoidal functors

$$\omega_1 : \mathscr{C}_1 \cong \mathscr{C}_1 \times 0 \xrightarrow{F} {}_H\mathbb{M} \xrightarrow{{}_HU} \mathrm{Vect}_k^{\mathrm{f.d.}},$$
$$\omega_2 : \mathscr{C}_2 \cong 0 \times \mathscr{C}_2 \xrightarrow{F} {}_H\mathbb{M} \xrightarrow{{}_HU} \mathrm{Vect}_k^{\mathrm{f.d.}}$$

are faithful and exact,

by Theorem 40 there are bialgebras A, B such that $G_1 : \mathscr{C}_1 \simeq \mathbb{M}^A$, $G_2 : \mathscr{C}_2 \simeq \mathbb{M}^B$ and $U^A \circ G_1 = \omega_1$, $U^B \circ G_2 = \omega_2$. Thus we have a k-linear monoidal equivalence

$$G : \mathbb{M}^{H^*} = {}_H\mathbb{M} \simeq \mathscr{C}_1 \times \mathscr{C}_2 \simeq \mathbb{M}^A \times \mathbb{M}^B \cong \mathbb{M}^{A \oplus B}$$

satisfying $U^{A\oplus B} \circ G = U^{H^*}$. By Theorem 39 there is a WBA isomorphism $\varphi : A \oplus B \longrightarrow H^*$ such that $G = \mathbb{M}^{\varphi}$. Therefore,

$$H \cong H^{**} \cong (A \oplus B)^* \cong A^* \oplus B^*$$

as WBAs. This is a contradiction. □

Let us recall Theorem 18: $D(H) = D(A) \oplus D(B)$ for the direct sum $H = A \oplus B$ of two finite-dimensional WHAs A and B.

Problem 41

Is it true that $\mathcal{Z}(\mathscr{C}_1 \times \mathscr{C}_2) \simeq \mathcal{Z}(\mathscr{C}_1) \times \mathcal{Z}(\mathscr{C}_2)$ for $\boldsymbol{k}$-linear monoidal categories $\mathscr{C}_1$ and $\mathscr{C}_2$?

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