On the Universal R-matrices of the Dihedral groups

(This is an English and modified version of "二面体群の普遍R行列について",
which is published in RIMS Kōkyuroku **1057** (1998), 41–53.)

Michihisa Wakui
(March 18, 2010)

In this note $\mathbb{C}$ denotes the complex number field, and $\mathbb{Z}_m$ denotes the cyclic group of order m.

Abstract : In this note we determine the universal R-matrices of the group $D_{m,n}$ generated by s and t subject to the relations $s^m = 1,\ t^2 = s^n$ and $tst^{-1} = s^{-1}$. Here, $m \geq 3$ and $n \geq 1$ are integers. The group $D_{m,n}$ is the dihedral group D_{2m} of order $2m$ if $m = n$, and is the quaternion group Q_{2m} of order $2m$ if $m = 2n$.

Theorem. *If $m \neq 4$, or if $m = 4$ and n is odd, then the universal R-matrices of $\mathbb{C}[D_{m,n}]$ are the universal R-matrices of $\mathbb{C}[\langle s\rangle]$, where $\langle s\rangle$ is the cyclic subgroup generated by s. So, the number of the universal R-matrices of $\mathbb{C}[D_{m,n}]$ is m.*

If $m = 4$ and n is even, then a universal R-matrix of $\mathbb{C}[D_{m,n}]$ is one of the following.
(i) *a universal R-matrix of $\mathbb{C}[\langle s\rangle]$,*
(ii) $\tilde{R}_{a,\mu} = \dfrac{1}{4}\displaystyle\sum_{\alpha,\beta,i,j=0,1} a^{\alpha\beta}(-1)^{j\alpha+i\beta}t^{\alpha}s^{2i+\alpha\mu} \otimes t^{\beta}s^{2j+\beta\mu}$, *where* $a^2 = (-1)^{\frac{n}{2}},\ \mu = 0, 1.$
So, the number of the universal R-matrices of $\mathbb{C}[D_{4,n}]$, where n is even, is 8.

In order to prove the theorem we use representation theory of cyclic groups.
As a corollary to Theorem we obtain the following:

Corollary. *The representation categories of $\mathbb{C}[D_8]$ and $\mathbb{C}[Q_8]$ are not equivalent as abstract tensor categories.*

We prove this by calculating the category-theoretic rank of the quasitriangular Hopf algebra $(\mathbb{C}[D_{4,n}],\ \tilde{R}_{a,\mu})$. The concept of category-theoretic rank is introduced by Majid [5].

Introduction. The notion of a universal R-matrix was introduced by Drinfel'd at the same time of finding quantum groups [3][4]. The notion of a ribbon Hopf algebra was introduced by Reshetikhin and Turaev to construct invariants of knots and 3-manifolds [9][10] together with a background of quantum physics. It is known that such invariants are one of the most powerful tools in low-dimensional topology. So, it is important to find a new solution of a universal R-matrix and a ribbon element of a given Hopf algebra.

For integers $m \geq 3$ and $n \geq 1$, let $D_{m,n}$ denotes the finite group defined by

$$D_{m,n} = \langle s,\ t \mid s^m = 1,\ t^2 = s^n,\ tst^{-1} = s^{-1}\rangle,$$

which includes a cyclic subgroup of index 2. If $m = n$, then $D_{m,n}$ is the dihedral group D_{2m} of order $2m$, and if $m = 2n$, then it is the quaternion group Q_{2m} of order $2m$. We determine the universal R-matrices of the group Hopf algebra $\mathbb{C}[D_{m,n}]$, and also determine the ribbon elements of any quasitriangular Hopf algebra $(\mathbb{C}[D_{m,n}], R)$ by using the representation theory of cyclic groups. On general facts of Hopf algebras and representation theory, see the Abe's book [1] and the Curtis and Reiner's book [2], respectively.

Let us recall the definitions of a universal R-matrix and a ribbon element.

Definition(**Drinfel'd [2]**) Let $A = (A, \Delta, \varepsilon, S)$ be a Hopf algebra over a field $\boldsymbol{k}$, and let R be an invertible element in $A \otimes A$. The pair (A, R) is said to be a *quasitriangular Hopf algebra*, or R is said to be a *universal R-matrix* of A if the following conditions are satisfied:

(QT.1) $\Delta^{\mathrm{cop}}(a) \cdot R = R \cdot \Delta(a)$ for all $a \in A$,
(QT.2) $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$,
(QT.3) $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.

Here, Δ^{cop} denotes the opposite comultiplication defined by $\Delta^{\mathrm{cop}} = T \circ \Delta$, and $R_{ij} \in A \otimes A \otimes A$ is given by $R_{12} = R \otimes 1, R_{23} = 1 \otimes R$ and $R_{13} = (T \otimes \mathrm{id})(R_{23})$, and T is the linear transformation on $A \otimes A$ such that $T(a \otimes b) = b \otimes a$ for all $a, b \in A$.

It is immediate to show that for a universal R-matrix R, the equations $(\varepsilon \otimes \mathrm{id})(R) = 1$, $(\mathrm{id} \otimes \varepsilon)(R) = 1$ hold. Conversely, by Radford [7, p.4 Lemma 1], if an element $R \in A \otimes A$ satisfies the conditions

(i) $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$,
(ii) $(\varepsilon \otimes \mathrm{id})(R) = 1$,

then R is invertible. Thus, an element $R \in A \otimes A$ is a universal R-matrix of A if and only if the conditions (QT.1), (QT.2), (QT.3) and

(QT.4) $(\varepsilon \otimes \mathrm{id})(R) = 1$

are satisfied.

For a finite group G the group algebra $\boldsymbol{k}[G]$ over a field $\boldsymbol{k}$ has a canonical Hopf algebra structure. The structure is given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \quad \text{and} \quad S(g) = g^{-1} \quad \text{for } g \in G.$$

This Hopf algebra is referred as the *group Hopf algebra* of G over $\boldsymbol{k}$. Any group Hopf algebra has at least one universal R-matrix, namely, $R = 1 \otimes 1$.

For a quasitriangular Hopf algebra $(A, R = \sum_i \alpha_i \otimes \beta_i)$, a distinguished element $u \in A$, called the *Drinfel'd element* of (A, R), is defined by

$$u = \sum_i S(\beta_i)\alpha_i.$$

The Drinfel'd element u is invertible, and $S^2(a) = uau^{-1}$ for all $a \in A$.

Definition.(**Reshetikhin and Turaev [9]**) Let (A, R) be a quasitriangular Hopf algebra over a field $\boldsymbol{k}$, and u be the Drinfel'd element of (A, R). A central element $v \in A$ is said to be a *ribbon element*, or the triple (A, R, v) is said to be a *ribbon Hopf algebra* if the following four conditions are satisfied:

(Rib.1) $v^2 = uS(u)$,
(Rib.2) $S(v) = v$,
(Rib.3) $\varepsilon(v) = 1$,
(Rib.4) $\Delta(v) = (R_{21}R)^{-1}(v \otimes v)$,

where $R_{21} = \sum_i \beta_i \otimes \alpha_i$ for $R = \sum_i \alpha_i \otimes \beta_i$.

1. The Main results

To describe our main results, we need to the universal R-matrices and the ribbon elements of the group Hopf algebra of a cyclic group. The following theorem is partially obtained by H.Murakami, Ohtsuki, Okada [6], and Radford [7, 8].

Theorem 1.1. *Let s be a generator of $\mathbb{Z}_m$, and ζ be a primitive m-th root of unity. Then, the universal R-matrices of the group Hopf algebra $\mathbb{C}[\mathbb{Z}_m]$ are given by the formula*

$$R_d = \frac{1}{m} \sum_{k,i=0}^{m-1} \zeta^{-ik} s^k \otimes s^{di} \qquad (d \in \{0, 1, \cdots, m-1\}).$$

Furthermore, the number of the ribbon elements of the quasitriangular Hopf algebra $(\mathbb{C}[\mathbb{Z}_m], R_d)$ is one or two according to the parity of m. More precisely, the ribbon elements are given by the formula

$$v = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \nu^j \zeta^{-dj^2 - kj} s^k,$$

where $\nu = 1$ if m is odd, and $\nu = \pm 1$ if m is even.

A proof of this theorem is given in the next section.

Now, we describe the main results. Let $m \geq 3$ and $n \geq 1$ be integers. In the group $D_{m,n}$, the subgroup generated by s is isomorphic to the cyclic group $\mathbb{Z}_m$. Since all universal R-matrices R_d $(d = 0, 1, \cdots, m-1)$ in Theorem 1.1 satisfy $\Delta^{\mathrm{cop}}(t) \cdot R_d = R_d \cdot \Delta(t)$, they are also universal R-matrices of $\mathbb{C}[D_{m,n}]$. Furthermore, the ribbon elements v given in Theorem 1.1 satisfy $vt = tv$. Thus, they are ribbon elements of the quasitriangular Hopf algebra $(\mathbb{C}[D_{m,n}], R_d)$.

Theorem 1.2. *If $m \neq 4$, or if $m = 4$ and n is odd, then the number of universal R-matrices of $\boldsymbol{C}[D_{m,n}]$ is m, and a universal R-matrix of $\boldsymbol{C}[D_{m,n}]$ is a universal R-matrix of $\boldsymbol{C}[\langle s \rangle]$. A ribbon element of the quasitriangular Hopf algebra $(\mathbb{C}[D_{m,n}], R_d)$ is a ribbon* *element of* $(\mathbb{C}[\langle s\rangle], R_d)$. *If* $m = 4$ *and* n *is even, then a universal R-matrix of* $\boldsymbol{C}[D_{m,n}]$ *is one of the universal R-matrices* R_d $(d = 0, 1, 2, 3)$*, and*

$$\tilde{R}_{a,\mu} = \frac{1}{4} \sum_{\alpha,\beta,i,j=0,1} a^{\alpha\beta}(-1)^{j\alpha+i\beta} t^{\alpha} s^{2i+\alpha\mu} \otimes t^{\beta} s^{2j+\beta\mu},$$

where $a^2 = (-1)^{\frac{n}{2}}$ *and* $\mu = 0, 1$. *If* $a^2 = 1$*, then the ribbon elements of* $(\mathbb{C}[D_{4,n}], \tilde{R}_{a,\mu})$ *are* $1,\ s^2$*, and if* $a^2 = -1$*, then the ribbon elements of* $(\mathbb{C}[D_{4,n}], \tilde{R}_{a,\mu})$ *are* $\pm\frac{\sqrt{-1}}{2}(1 - s^2) + \frac{1}{2}(s + s^3)$.

The proof of this theorem is also given in the next section.

2. Proofs of Theorems 1.1 and 1.2

For a finite group G let $\chi_1, \cdots, \chi_n$ be the irreducible characters of G over $\mathbb{C}$. For each $i = 1, \cdots, n$, we set

$$E_i := \frac{\deg \chi_i}{|G|} \sum_{g\in G} \chi_i(g^{-1})g \quad \in \mathbb{C}[G].$$

Then, E_i $(i = 1, \cdots, n)$ are the primitive idempotents of $\mathbb{C}[G]$, that is,

$$E_iE_j = \delta_{ij}E_i \quad (i, j = 1, \cdots, n), \quad E_1 + \cdots + E_n = 1.$$

Let s be a generator of the cyclic group $\mathbb{Z}_m$, and ζ be a primitive m-th root of unity. Then, the irreducible characters $\chi_k : \mathbb{Z}_m \longrightarrow \mathbb{C}$ $(k = 0, 1, \cdots, m - 1)$ are given by

$$\chi_k(s^i) = \zeta^{ki} \quad (i = 0, 1, \cdots, m - 1).$$

Thus, the primitive idempotent E_k associated to the irreducible character χ_k is given by

$$E_k = \frac{1}{m} \sum_{i=0}^{m-1} \zeta^{-ki} s^i. \quad \cdots\cdots\cdots \text{①}$$

Since $E_k s = \zeta^k E_k$, the equation

$$s^i = 1 \cdot s^i = \sum_{k=0}^{m-1} E_k s^i = \sum_{k=0}^{m-1} \zeta^{ik} E_k$$

holds for $i = 0, 1, \cdots, m - 1$.

For a convenience, we define E_k for all integer k by the right-hand side of ①. Then, the equation $E_{m+k} = E_k$ holds for all $k \in \mathbb{Z}$. Since ζ is a primitive m-th root of unity, for an integer p,

$$\sum_{i=0}^{m-1} \zeta^{ip} = \begin{cases} m & \text{if } p \equiv 0 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\Delta(E_k) = \sum_{j=0}^{m-1} E_j \otimes E_{k-j}$.

(**Proof of Theorem 1.1**) First, let us derive the necessary and sufficient conditions that an element $R \in \mathbb{C}[\mathbb{Z}_m] \otimes \mathbb{C}[\mathbb{Z}_m]$ is a universal R-matrix. We write R in the from

$$R = \sum_{i,j=0}^{m-1} a_{ij} E_i \otimes E_j \quad (a_{ij} \in \mathbb{C}),$$

where the indices i, j of a are treated as modulo m. Then, we have

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$$
$$\iff a_{i+k\ j} = a_{kj}a_{ij} \quad \text{for} \quad i, j, k = 0, 1, \cdots, m-1$$
$$\implies \begin{cases} a_{ij} = (a_{1j})^i \text{ for } \ i = 1, \cdots, m-1 \text{ and } j = 0, 1, \cdots, m-1, \\ a_{0j} = (a_{1j})^m. \end{cases}$$

Similarly, we have

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$$
$$\implies \begin{cases} a_{ij} = (a_{i1})^j \text{ for } \ i = 0, 1, \cdots, m-1 \text{ and } j = 1, \cdots, m-1, \\ a_{i0} = (a_{i1})^m. \end{cases}$$

Thus, if R is a universal R-matrix, then $a_{1j} = (a_{11})^j$ is required for $j = 1, \cdots, m-1$. Since $\varepsilon(E_k) = \frac{1}{m}\sum_{i=0}^{m-1} \zeta^{-ki} = \delta_{k0}$, we see that

$$\begin{cases} (\varepsilon \otimes \mathrm{id})(R) = 1 \iff a_{0j} = 1 \quad (j = 0, 1, \cdots, m-1), \\ (\mathrm{id} \otimes \varepsilon)(R) = 1 \iff a_{i0} = 1 \quad (i = 0, 1, \cdots, m-1). \end{cases}$$

Hence, if R is a universal R-matrix of $\mathbb{C}[\mathbb{Z}_m]$, then

$$a_{11}^m = 1 \text{ and } a_{ij} = (a_{11})^{ij} \quad (i, j = 0, 1, \cdots, m-1).$$

From $a_{11}^m = 1$, the number a_{11} can be expressed as $a_{11} = \zeta^d$ for some $d \in \{0, 1, \cdots, m-1\}$. Then,

$$\begin{aligned} R &= \sum_{i,j=0}^{m-1} \zeta^{dij} E_i \otimes E_j \\ &= \frac{1}{m^2} \sum_{l,k,i=0}^{m-1} \zeta^{-ik} \sum_{j=0}^{m-1} \zeta^{dij-jl} s^k \otimes s^l \\ &= \frac{1}{m} \sum_{k,i=0}^{m-1} \zeta^{-ik} s^k \otimes s^{di}. \end{aligned}$$

Conversely, it can be checked that the above R is a universal R-matrix of $\mathbb{C}[\mathbb{Z}_m]$.

Next, we show the remaining part. For the above R, the Drinfel'd element u is given by

$$u = \frac{1}{m} \sum_{k,i=0}^{m-1} \zeta^{-ik} s^{-k+di} = \frac{1}{m} \sum_{i,j=0}^{m-1} \zeta^{dij} \sum_{k=0}^{m-1} \zeta^{-k(i+j)} E_j = \sum_{j=0}^{m-1} \zeta^{-dj^2} E_j.$$

Since $S(E_j) = E_{-j}$ $(j = 0, 1, \cdots, m-1)$, we have $uS(u) = \sum_{j=0}^{m-1} \zeta^{-2dj^2} E_j$. Thus, a ribbon element v of the quasitriangular Hopf algebra $(\mathbb{C}[\mathbb{Z}_m], R)$ can be written as in the form

$$v=\sum_{j=0}^{m-1}\nu_j\zeta^{-dj^2}E_j,$$

where $\nu_j=\pm 1$ $(j=0,1,\cdots,m-1)$. Hereinafter, the index j of ν is treated as modulo m. Since

$$S(v)=\sum_{j=0}^{m-1}\nu_{m-j}\zeta^{-dj^2}E_j,\quad \varepsilon(v)=\sum_{j=0}^{m-1}\nu_j\zeta^{-dj^2}\delta_{j0}=\nu_0,$$

we have

$$\begin{cases} S(v)=v \iff \nu_j=\nu_{m-j} \quad (j=1,\cdots,m-1), & \cdots\cdots\cdots ② \\ \varepsilon(v)=1 \iff \nu_0=1. & \cdots\cdots\cdots ③ \end{cases}$$

Since

$$\begin{aligned} R_{21}R=&\frac{1}{m^2}\sum_{i,j,k,l=0}^{m-1}\zeta^{-jk-il}s^{k+di}\otimes s^{l+dj} \\ =&\frac{1}{m^2}\sum_{a,b=0}^{m-1}\sum_{i,j=0}^{m-1}\zeta^{dia+djb}\sum_{k=0}^{m-1}\zeta^{(-j+a)k}\sum_{l=0}^{m-1}\zeta^{(-i+b)l}E_a\otimes E_b \\ =&\sum_{a,b=0}^{m-1}\zeta^{2dab}E_a\otimes E_b \end{aligned}$$

and

$$\Delta(v)=\sum_{a=0}^{m-1}\sum_{b=0}^{m-1}\nu_{a+b}\zeta^{-d(a+b)^2}E_a\otimes E_b,$$

we have

$$\begin{aligned} (R_{21}R)\cdot\Delta(v)=v\otimes v &\iff \nu_a\nu_b=\nu_{a+b}\ (a,b=0,1,\cdots,m-1) \\ &\iff \nu_a=\nu_1^a \quad (a=1,\cdots,m-1,m) \quad \text{or} \\ &\qquad \nu_a=0 \quad (a=0,1,\cdots,m-1). \qquad \cdots\cdots ④ \end{aligned}$$

It follows from ②,③,④ that $\nu_1^m=1$, and hence, if m is odd, then $\nu_1=1$. Thus, if v is a ribbon element of the quasitriangular Hopf algebra $(\mathbb{C}[\mathbb{Z}_m],R_d)$, then v is given by

$$v=\sum_{j=0}^{m-1}\nu_1^j\zeta^{-dj^2}E_j,$$

where $\nu_1=1$ if m is odd, and $\nu_1=\pm 1$ if m is even. Conversely, the above v satisfies ②,③,④, and hence v is a ribbon element of $(\mathbb{C}[\mathbb{Z}_m],R_d)$. □

Let us concern to prove Theorem 1.2. Since the cyclic subgroup $\langle s\rangle$ of $D_{m,n}$ has order m, the subgroup can be identified with $\mathbb{Z}_m$. Then, the set $\{E_0,E_1,\cdots,E_{m-1}\}\cup\{tE_0,tE_1,\cdots,tE_{m-1}\}$ is a basis of $\mathbb{C}[D_{m,n}]$. By using this basis, the universal R-matrices and the ribbon elements of $\mathbb{C}[D_{m,n}]$ are determined .

(**Proof of Theorem 1.2**) Let us determine the universal R-matrices of $\mathbb{C}[D_{m,n}]$. Let R be an element of $\mathbb{C}[D_{m,n}] \otimes \mathbb{C}[D_{m,n}]$, and write it as in the form

$$R = \sum_{\substack{\alpha,\beta=0,1\\ i,j=0,1,\cdots,m-1}} a_{\beta j}^{\alpha i}\ t^\alpha E_i \otimes t^\beta E_j \quad (a_{\beta j}^{\alpha i} \in \mathbb{C}),$$

where the indices i, j of $a_{\beta j}^{\alpha i}$ are treated as modulo m. Since

$$\begin{aligned} R \cdot s \otimes s &= \sum_{\substack{\alpha,\beta\\ i,j}} a_{\beta j}^{\alpha i} \zeta^{i+j} t^\alpha E_i \otimes t^\beta E_j, \\ s \otimes s \cdot R &= \sum_{\substack{\alpha,\beta\\ i,j}} a_{\beta j}^{\alpha i} t^\alpha s^{(-1)^\alpha} E_i \otimes t^\beta s^{(-1)^\beta} E_j \\ &= \sum_{\substack{\alpha,\beta\\ i,j}} a_{\beta j}^{\alpha i} \zeta^{(-1)^\alpha i + (-1)^\beta j} t^\alpha E_i \otimes t^\beta E_j, \end{aligned}$$

we have

$$s \otimes s \cdot R = R \cdot s \otimes s \iff a_{\beta j}^{\alpha i} \zeta^{i+j} = a_{\beta j}^{\alpha i} \zeta^{(-1)^\alpha i + (-1)^\beta j} \quad \text{for all } \alpha, \beta, i, j.$$

Here, for $\alpha, \beta = 0, 1$,

$$\begin{aligned} \zeta^{i+j} = \zeta^{(-1)^\alpha i + (-1)^\beta j} &\iff i + j \equiv (-1)^\alpha i + (-1)^\beta j \bmod m \\ &\iff \begin{cases} 2i \equiv 0 \pmod m & \text{if } \alpha = 1,\ \beta = 0, \\ 2j \equiv 0 \pmod m & \text{if } \alpha = 0,\ \beta = 1, \\ 2(i+j) \equiv 0 \pmod m & \text{if } \alpha = 1,\ \beta = 1. \end{cases} \end{aligned}$$

Thus, under the assumption $m \geq 3$, we have:

- $a_{0j}^{11} = a_{11}^{0i} = 0$ for $i, j = 0, 1, \cdots, m-1$, $\cdots\cdots\cdots$ ❶
- in the case when m is odd, $a_{1j}^{11} = a_{11}^{1j} = 0$ for $j \neq m-1$, $\cdots\cdots\cdots$ ❷
- in the case when m is even, $a_{1j}^{11} = a_{11}^{1j} = 0$ for $j \neq m'-1, m-1$.

On the other hand, since $E_k t = t E_{-k}$ for $k = 0, 1, \cdots, m-1$, we have

$$t \otimes t \cdot R = R \cdot t \otimes t \iff a_{\beta j}^{\alpha i} = a_{\beta\ -j}^{\alpha\ -i} \quad \text{for all } \alpha, \beta, i, j. \quad \cdots\cdots\cdots ❸$$

Since

$$\begin{aligned} (\Delta \otimes \mathrm{id})(R) &= \sum_{\substack{\alpha,\beta\\ i,j}} \sum_k a_{\beta j}^{\alpha i}\ t^\alpha E_k \otimes t^\alpha E_{i-k} \otimes t^\beta E_j, \\ R_{13} R_{23} &= \sum_{\substack{\alpha,\beta,\alpha',\beta'\\ i,j,k,l}} a_{\beta l}^{\alpha k} a_{\beta' j}^{\alpha' i}\ t^\alpha E_k \otimes t^{\alpha'} E_i \otimes t^{\beta+\beta'} E_{(-1)^{\beta'} l} E_j \\ &= \sum_{\substack{\alpha,\alpha',\beta\\ i,j,k}} \sum_{\beta_1+\beta_2=\beta} a_{\beta_1 (-1)^{\beta_2} j}^{\alpha\quad k} a_{\beta_2 j}^{\alpha' i} \zeta^{nj\beta_1\beta_2}\ t^\alpha E_k \otimes t^{\alpha'} E_i \otimes t^\beta E_j, \end{aligned}$$

a necessary and sufficient condition for $(\Delta \otimes \mathrm{id})(R) = R_{13} R_{23}$ is

$$\delta_{\alpha,\alpha'} a_{\beta\ j}^{\alpha\ i+k} = \sum_{\beta_1+\beta_2=\beta} a_{\beta_1 (-1)^{\beta_2} j}^{\alpha\quad k} a_{\beta_2 j}^{\alpha' i} \zeta^{nj\beta_1\beta_2} \quad \cdots\cdots\cdots ❹$$

for all $\alpha, \alpha', \beta = 0, 1,\ i, j, k = 0, 1, \cdots, m-1$. Here, $\delta_{\alpha,\alpha'}$ stands for Kronecker's delta.

Similarly, we see that a necessary and sufficient condition for $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$ is

$$\delta_{\beta,\beta'} a_{\beta\ j+k}^{\alpha\ i} = \sum_{\alpha_1+\alpha_2=\alpha} a_{\beta\ \ j}^{\alpha_1 (-1)^{\alpha_2} i} a_{\beta' k}^{\alpha_2 i} \zeta^{ni\alpha_1\alpha_2} \qquad \cdots\cdots\cdots ❹'$$

for all $\alpha, \beta, \beta' = 0, 1,\ i, j, k = 0, 1, \cdots, m-1$. By using ❹ and ❹$'$, repeatedly, we have

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$$

$$\Longrightarrow \quad \text{for all } \alpha, \beta = 0, 1,\ i, j = 0, 1, \cdots, m-1$$

$$a_{\beta\ j}^{\alpha\ i+1} = \sum_{\beta_1+\beta_2+\cdots+\beta_{i+1}=\beta} \Big(\prod_{g=1}^{i} a_{\beta_g (-1)^{\beta_{g+1}+\cdots+\beta_{i+1}} j}^{\alpha\ \ 1}\Big) a_{\beta_{i+1} j}^{\alpha\ \ 1} \zeta^{nj \sum_{u<v} \beta_u \beta_v},$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$$

$$\Longrightarrow \quad \text{for all } \alpha, \beta = 0, 1,\ i, j = 0, 1, \cdots, m-1$$

$$a_{\beta\ 1+k}^{\alpha\ i} = \sum_{\alpha_1+\alpha_2+\cdots+\alpha_{k+1}=\alpha} \Big(\prod_{g=1}^{k} a_{\beta\ \ \ 1}^{\alpha_g\ (-1)^{\alpha_{g+1}+\cdots+\alpha_{k+1}} i}\Big) a_{\beta\ \ 1}^{\alpha_{k+1} i} \zeta^{ni \sum_{u<v} \alpha_u \alpha_v}.$$

By ❶, we obtain

$$(2.3) \qquad \begin{cases} a_{\beta\ j}^{1\ i+1} = & \delta_{\beta,i+1} (a_{1\ j}^{1\ 1})^{[\frac{i}{2}]+1} (a_{1\ -j}^{1\ 1})^{[\frac{i+1}{2}]} \zeta^{nji(i+1)/2}, \\ a_{1\ 1+k}^{\alpha\ i} = & \delta_{\alpha,k+1} (a_{1\ 1}^{1\ i})^{[\frac{k}{2}]+1} (a_{1\ 1}^{1\ -i})^{[\frac{k+1}{2}]} \zeta^{nik(k+1)/2}, \end{cases}$$

where the indices of Kronecker's deltas $\delta_{\beta,i+1},\ \delta_{\alpha,k+1}$ are treated as modulo 2, and for a real number x, denoted by $[x]$ is the maximum integer such that it is less than or equal to x. By (2.3) and ❷, in the case when m is odd, we have

$$\begin{cases} a_{\beta\ j}^{1\ i+1} = 0 & \text{for } \beta = 0, 1 \text{ and } (i, j) \neq (0, m-1), \\ a_{1\ 1+k}^{\alpha\ i} = 0 & \text{for } \alpha = 0, 1 \text{ and } (k, i) \neq (0, m-1), \end{cases}$$

and in the case when $m \neq 4$ is even, by $1 \neq m'-1, m-1,\ m'+1 \neq m'-1, m-1$, we have

$$\begin{cases} a_{\beta\ j}^{1\ i+1} = 0 & \text{for } \beta = 0, 1 \text{ and } (i, j) \neq (0, m-1), (0, m'-1), \\ a_{1\ 1+k}^{\alpha\ i} = 0 & \text{for } \alpha = 0, 1 \text{ and } (k, i) \neq (0, m-1), (0, m'-1). \end{cases}$$

Furthermore, by ❸ we see that $a_{\beta\ m-1}^{1\ 1} = a_{\beta\ 1}^{1\ m-1}$, $a_{\beta\ m'-1}^{1\ 1} = a_{\beta\ m'+1}^{1\ m-1}$ and $a_{1\ 1}^{\alpha\ m-1} = a_{1\ m-1}^{\alpha\ 1}$, $a_{1\ 1}^{\alpha\ m'-1} = a_{1\ m-1}^{\alpha\ m'+1}$. So, if $m \neq 4$, then

$$\begin{cases} a_{\beta\ j}^{1\ i} = 0 & \text{for } \beta = 0, 1 \text{ and } i, j = 0, 1, \cdots, m-1, \\ a_{1\ j}^{\alpha\ i} = 0 & \text{for } \alpha = 0, 1 \text{ and } i, j = 0, 1, \cdots, m-1. \end{cases}$$

This concludes that if $m \neq 4$, then any universal R-matrix of $\mathbb{C}[D_{m,n}]$ is a universal R-matrix of $\mathbb{C}[\mathbb{Z}_m]$.

Let us consider the case when $m = 4$. By ❶, the equations $a_{0i}^{11} = a_{11}^{0i} = 0$ $(i = 0, 1, 2, 3)$ hold. Combining these equations and ❸, we have $a_{0i}^{13} = a_{13}^{0i} = 0$ $(i = 0, 1, 2, 3)$. On the other hand, by ❷, the equation $a_{10}^{11} = a_{12}^{11} = a_{11}^{10} = a_{11}^{12} = 0$ hold. Combining these equations and (2.1), we have $a^{1i}_{10} = a^{1i}_{12} = a^{10}_{1i} = a^{12}_{1i} = 0$ $(i = 0, 1, 2, 3)$. Moreover, by (2.3), we have

$$\begin{cases} a^{13}_{11} = a^{11}_{13}(a^{11}_{11})^2\zeta^{3n}, & a^{13}_{13} = a^{11}_{11}(a^{11}_{13})^2\zeta^{9n}, \\ a^{10}_{0j} = (a^{1\ 1}_{1\ j})^2(a^{1\ 1}_{1-j})^2\zeta^{6nj}, & a^{12}_{0j} = a^{1\ 1}_{1\ j}a^{1\ 1}_{1-j}\zeta^{nj}, \\ a^{0i}_{10} = (a^{1\ i}_{1\ 1})^2(a^{1-i}_{1\ \ 1})^2\zeta^{6ni}, & a^{0i}_{12} = a^{1\ i}_{1\ 1}a^{1-i}_{1\ \ 1}\zeta^{ni}. \end{cases} \tag{2.4}$$

We set $a := a^{11}_{13} = a^{13}_{11}$.

• If $a = 0$, then by (2.4) and ❸,

$$a^{1i}_{1j} = a^{1i}_{0j} = a^{0i}_{1j} = 0$$

for all $i, j = 0, 1, 2, 3$. Thus, in this case, R is a universal R-matrix of $\mathbb{C}[\langle s\rangle]$.

• If $a \neq 0$, then by (2.4),

$$\begin{cases} (a^{11}_{11})^2 = \zeta^{-3n} = \zeta^n, & & 1 = a^2\zeta^{9n} = a^2\zeta^n, \\ a^{10}_{01} = a^2\zeta^{-n}, & a^{12}_{01} = a^{11}_{11}a\zeta^n, & a^{12}_{03} = a^{11}_{11}a\zeta^{-n}, \\ a^{01}_{10} = a^2\zeta^{-n}, & a^{01}_{12} = a^{11}_{11}a\zeta^n, & a^{03}_{12} = a^{11}_{11}a\zeta^{-n}. \end{cases}$$

Combining these equations and ❸, we see that $\zeta^{2n} = 1,\ a^2 = \zeta^n$. In particular, n is needed to be even. So, for $\nu = \pm 1$, we have

$$a^{1i}_{1j} = \begin{cases} a & \text{if } (i,j) = (1,3),\ (3,1), \\ \nu a & \text{if } (i,j) = (1,1),\ (3,3), \\ 0 & \text{otherwise,} \end{cases}$$

$$a^{1i}_{0j} = a^{0j}_{1i} = \begin{cases} 1 & \text{if } (i,j) = (0,1),\ (0,3), \\ \nu & \text{if } (i,j) = (2,1),\ (2,3), \\ 0 & \text{otherwise.} \end{cases}$$

To determine the value of a^{0i}_{0j}, we substitute $\alpha' = 0,\ \alpha = \beta = 1,\ i = j = k = 1$ to ❹. Then

$$0 = \sum_{\beta_1+\beta_2=1} a^{1\quad\ \ 1}_{\beta_1(-1)^{\beta_2}} a^{0\ 1}_{\beta_2 1}\zeta^{n\beta_1\beta_2}.$$

Since $a^{01}_{11} = 0$, the above equation is equivalent to $a^{11}_{11}a^{01}_{01} = 0$. Since $a^{11}_{11} \neq 0$, it follows that $a^{01}_{01} = 0$. From (2.1) and (2.2) in the case of $\alpha = \beta = 0$, the equations $a^{0i}_{01} = a^{01}_{0i} = 0$ $\ (i = 0, 1, 2, 3)$ are obtained. Thus, by ❸, the equations $a^{0i}_{03} = a^{03}_{0i} = 0$ $\ (i = 0, 1, 2, 3)$ hold. From (2.1) and (2.2) in the case of $\alpha = \beta = 0$ and $a^{01}_{00} = a^{01}_{02} = 0$, we have $a^{00}_{00} = a^{02}_{00} = a^{00}_{02} = a^{02}_{02} = 1$. As a consequence,

$$a^{0i}_{0j} = \begin{cases} 1 & \text{if } (i,j) = (0,0), (0,2), (2,0), (2,2), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $a \neq 0$ (and n is even), then R can be written as

$$R = \sum_{\substack{\alpha,\beta=0,1 \\ i,j=0,1}} a^{\alpha\beta}\nu^{i\alpha+j\beta+\alpha\beta}t^\alpha E_{2i+\beta} \otimes t^\beta E_{2j+\alpha}, \tag{2.5}$$

where $a^2 = (-1)^{\frac{n}{2}}$ and $\nu = \pm 1$. It is easy to show that the above R satisfies $(\varepsilon \otimes \mathrm{id})(R) = 1$. Since

$$E_0 + E_2 = \frac{1}{2}(s^0 + s^2), \quad E_1 + E_3 = \frac{1}{2}(s^0 - s^2),$$
$$E_0 - E_2 = \frac{1}{2}(s + s^3), \quad E_1 - E_3 = \frac{\sqrt{-1}}{2}(-s + s^3),$$

the element R of (2.5) can be rewritten as

$$R = \frac{1}{4} \sum_{\substack{\alpha,\beta=0,1 \\ i,j=0,1}} a^{\alpha\beta}(-1)^{j\alpha+i\beta} t^\alpha s^{2i+\alpha\mu} \otimes t^\beta s^{2j+\beta\mu},$$

where $\mu = 0$ or $\mu = 1$ according to $\nu = 1$ or $\nu = -1$. Conversely, one can prove that the above R is a universal R-matrix of $\mathbb{C}[D_{4,n}]$.

Let us determine when an element

$$v = \sum_{i=0}^{m-1} (a_i E_i + b_i t E_i), \qquad (a_i,\ b_i \in \mathbb{C},\ i = 0, 1, \cdots, m-1)$$

is a ribbon element of $(\mathbb{C}[D_{m,n}], R)$. Since

$$sv = \sum_{i=0}^{m-1} (a_i \zeta^i E_i + b_i \zeta^{-i} t E_i), \quad vs = \sum_{i=0}^{m-1} (a_i \zeta^i E_i + b_i \zeta^i t E_i),$$

we see that

$$vs = sv \iff b_i \zeta^{-i} = b_i \zeta^i \quad \text{for } i = 0, 1, \cdots, m-1.$$

For a given $i = 0, 1, \ldots, m-1$, a necessary and sufficient condition for $\zeta^i = \zeta^{-i}$ is $2i \equiv 0 \pmod{m}$. Thus, if $vs = sv$, then $b_1 = 0$. Next, let us consider a condition for $(R_{21}R)\Delta(v) = v \otimes v$. Note that $\Delta(v)$ is given by

$$\Delta(v) = \sum_{i,j=0}^{m-1} (a_i E_j \otimes E_{i-j} + b_i t E_j \otimes t E_{i-j})$$

under the treatment of the indices i, j of a and b as modulo m.

• In the case of $R := R_d$, we have

$$(R_{21}R)\Delta(v) = \sum_{i,j=0}^{m-1} (\zeta^{2dij} a_{i+j} E_i \otimes E_j + \zeta^{2dij} b_{i+j} t E_i \otimes t E_j).$$

So, for all $i, j = 0, 1, \cdots, m-1$,

$$(R_{21}R)\Delta(v) = v \otimes v \iff \begin{cases} a_i a_j = \zeta^{2dij} a_{i+j}, \\ a_i b_j = 0, \\ b_i b_j = \zeta^{2dij} b_{i+j}. \end{cases}$$

In particular, for each $i = 0, 1, \cdots, m-1$ the equation $b_{i+1} = \zeta^{-2di} b_i b_1$ is required. If v is a ribbon element, then $b_i = 0$ for all $i = 0, 1, \cdots, m-1$.

• In the case when $m = 4$, n is even, and $R = \tilde{R}_{a,\mu}$ $(a^2 = (-1)^{\frac{n}{2}}, \mu = 0, 1)$, then by (2.5)

$$R_{21}R$$
$$= \sum_{\substack{\alpha,\beta\gamma,\delta \\ i,j,k,l}} a^{\alpha\beta+\gamma\delta}\nu^{i\alpha+j\beta+\alpha\beta+k\gamma+l\delta+\gamma\delta}t^{\beta}E_{2j+\alpha}t^{\gamma}E_{2k+\delta}\otimes t^{\alpha}E_{2i+\beta}t^{\delta}E_{2l+\gamma}$$
$$= \sum_{\substack{\alpha,\beta\gamma,\delta \\ i,j,k,l}} a^{\alpha\beta+\gamma\delta}\nu^{i\alpha+j\beta+\alpha\beta+k\gamma+l\delta+\gamma\delta}t^{\beta+\gamma}E_{(-1)^{\gamma}(2j+\alpha)}E_{2k+\delta}\otimes t^{\alpha+\delta}E_{(-1)^{\delta}(2i+\beta)}E_{2l+\gamma}$$
$$= \sum_{\substack{\alpha,\beta=0,1 \\ i,j=0,1}} a^{2(i\alpha+j\beta)}E_{2i+\beta}\otimes E_{2j+\alpha}.$$

It follows that, in the expression of $(R_{21}R)\Delta(v)$ with respect to the basis $\{t^{\alpha}E_i \otimes t^{\beta}E_j \mid \alpha=0,1,\ i,j=0,1,2,3\}$, the coefficients of $tE_i\otimes E_j$ and $E_i\otimes tE_j$ are all 0. This implies that if v is a ribbon element, then $b_j=0$ for all $j=0,1,2,3$ since $a_0=1$ by the condition $\varepsilon(v)=1$. Under this condition, we have

$$(2.6)\qquad (R_{21}R)\Delta(v)=v\otimes v \iff a_{2i+\beta}a_{2j+\alpha}=a^{2(i\alpha+j\beta)}a_{2i+2j+\alpha+\beta}\ \text{ for all } i,j,\alpha,\beta.$$

By substituting $i=0,\ j=1,\ \alpha=0,\ \beta=1$ to the equation (2.6), we have $a_2=a^2=(-1)^{\frac{n}{2}}$, and by substituting $i=0,\ j=0,\ \alpha=1,\ \beta=1$ to the same equation (2.6), we have $a_2=a_1^2$. Furthermore, from the condition $S(v)=v$, the equation $a_1=a_3$ is obtained. Thus, any ribbon element v can be expressed as $v=E_0+a^2E_2\pm a(E_1+E_3)$. By writing it as a linear combination of $\{s^i\}$, we have the form in the theorem. Conversely, it is easy to check that the above v is a ribbon element of $(\mathbb{C}[D_{4,n}],\tilde{R}_{a,\mu})$ under the assumption $a^2=(-1)^{\frac{n}{2}}$. (Note that $u=\frac{1}{2}(1+s^2)+\frac{a}{2}(1-s^2)$.) □

3. An application

By comparing the universal R-matrices of the dihedral group Hopf algebra $\mathbb{C}[D_8]$ and the quaternion group Hopf algebra $\mathbb{C}[Q_8]$, that are determined in Theorem 1.2, we see that the representation categories of $\mathbb{C}[D_8]$ and $\mathbb{C}[Q_8]$ are not isomorphic as abstract tensor categories. To show this, we use the concept of the category-theoretic rank of a quasitriangular Hopf algebra, which was introduced by Majid [5].

Let (A,R) be a quasitriangular Hopf algebra, and u be the Drinfel'd element of (A,R). The trace of the left action of u on A is called the *rank* of (A,R), and it is denoted by $\mathrm{rank}(A,R)$.

Example 3.1. *Let n be an even integer. Then the rank of $(\mathbb{C}[D_{4,n}],R_d)$ is as follows:*

d	0	1	2	3
rank$(\mathbb{C}[D_{4,n}],R_d)$	8	$4(1-\sqrt{-1})$	0	$4(1+\sqrt{-1})$

In the case when $n=2,4$, for $\mu=0,1$ the rank of $(\mathbb{C}[D_{4,n}],\tilde{R}_{a,\mu})$ is as follows:

n	4		2	
(a,μ)	$(1,\mu)$	$(-1,\mu)$	$(\sqrt{-1},\mu)$	$(-\sqrt{-1},\mu)$
$rank(\mathbb{C}[D_{4,n}],\tilde{R}_{a,\mu})$	8	0	$4(1+\sqrt{-1})$	$4(1-\sqrt{-1})$

In general, if two representation categories of Hopf algebras A and B are tensor equivalent, then there exists a bijection between the sets of the universal R-matrices A and B such that it preserves representation-theoretic ranks. The above example shows that the universal R-matrices of $\mathbb{C}[D_8]$ and $\mathbb{C}[Q_8]$ are distinguished as (multi-)sets. This means that the two representation categories of $\mathbb{C}[D_8]$ and $\mathbb{C}[Q_8]$ are not equivalent as abstract tensor categories.

Acknowledgment. The author would like to thanks Professor Noriaki Kawanaka for helpful advice.

References

[1] E. Abe, *Hopf algebras*, Cambridge University Press, Cambridge, 1980 (original Japanese version published by Iwanami Shoten, Tokyo, 1977).

[2] C. W. Curtis and I. Reiner, *Methods of representation theory volume 1*, John Wiley & Sons, 1981.

[3] V. G. Drinfel'd, *Quantum groups*, in "Proceedings of the International Congress of Mathematics, Berkeley, CA., 1987" p.798—820.

[4] V. G. Drinfel'd, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990) p.321—342.

[5] S. Majid, *Representation-theoretic rank and double Hopf algebras*, Comm. Alg. **18** (1990) p.3705—3712.

[6] H. Murakami, T. Ohtsuki and M. Okada, *Invariants of three manifolds derived from linking matrices of framed links*, Osaka J. Math. **29** (1992) p.545—572.

[7] D. E. Radford, *On the antipode of a quasitriangular Hopf algebra*, J. of Alg. **151** (1992) p.1—11.

[8] D. E. Radford, *On Kauffman's knot invariants arising from finite-dimensional Hopf algebras*, in "Advances in Hopf algebras" (Lecture Notes in Pure and Applied Mathematics **158**), edited by J. Bergen and S. Montgomery, Marcel Dekker, 1994, p.205—266.

[9] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990) p.1—26.

[10] N. Yu. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991) p.547—597.

Note added after published.

• In the original Japanese version, there are mistakes in the statement of Theorem 4, which corresponds to Theorem 1.2 in this note, and its proof. The incorrect part in Theorem 4 is the statement on the ribbon elements of $(\mathbb{C}[D_{4,n}],\tilde{R}_{a,\mu})$ in the case of $a^2=-1$. In this note, this is corrected.

• For integers $m\geq 3$, $n\geq 1$ and q $(1\leq q\leq m-1)$, let us consider the finite group

$$D_{m,n,q}=\langle s,\ t\mid s^m=1,\ t^2=s^n,\ t^{-1}st=s^q\rangle.$$

This group can be regarded as a generalization of $D_{m,n}$. In particular, $D_{m,m,1}$ $(m \geq 3)$ is the direct product of the cyclic groups of order m and order 2, $D_{m,m,m-1}$ is the dihedral group D_{2m} of order $2m$:

$$D_{2m} = \langle s,\ t \mid s^m = 1,\ t^2 = 1,\ t^{-1}st = s^{-1}\rangle,$$

$D_{2n,n,2n-1}$ $(n \geq 2)$ is the generalized quaternion group Q_{4n} of order $4n$:

$$Q_{4n} = \langle s,\ t \mid s^{2n} = 1,\ t^2 = s^n,\ t^{-1}st = s^{-1}\rangle,$$

$D_{4p,4p,2p-1}$ $(p \geq 3)$ is the semi-dihedral group SD_{8p} of order $8p$:

$$SD_{8p} = \langle s,\ t \mid s^{4p} = 1,\ t^2 = 1,\ t^{-1}st = s^{2p-1}\rangle,$$

and $D_{4p,4p,2p+1}(p \geq 3)$ is the meta-abelian group SD_{8p} of order $8p$:

$$SA_{8p} := \langle s,\ t \mid s^{4p} = 1,\ t^2 = 1,\ t^{-1}st = s^{2p+1}\rangle.$$

By the same method of the proof of Theorem 1.2, one can determine the universal R-matrices of the group Hopf algebras for such a group.

If the order of the group $D_{m,n,q}$ is $2m$, then the subgroup of $D_{m,n,q}$ generated by s is a cyclic group of order m, and hence it can be identified with $\mathbb{Z}_m$.

Suppose that m, q satisfy the condition "for each $k \in \{0, 1, \cdots, m-1\}$ there exists a unique $j \in \{0, 1, \cdots, m-1\}$ such that $qj \equiv k \pmod{m}$". We write the integer j by $\sigma(k)$:

$$q\sigma(k) \equiv k \pmod{m}. \tag{A.1}$$

If the equations

$$\sigma(i)\sigma(k) = ikq \qquad \text{for all } i, k \in \{0, 1, \cdots, m-1\} \tag{A.2}$$

hold, then R_d $(d = 0, 1, \cdots, m-1)$ satisfies $\Delta^{\mathrm{cop}}(t) \cdot R_d = R_d \cdot \Delta(t)$, and hence it is a universal R-matrix of $\mathbb{C}[D_{m,n,q}]$. If $q = m-1$, or if $m = n = 4p$ and $q = 2p \pm 1$ (where $p \geq 3$), then the condition (A.2) is satisfied. Thus, R_d is a universal R-matrix of $\mathbb{C}[D_{m,n,q}]$.

Theorem. (i) For the group $D_{m,n,m-1}$ $(m \geq 3,\ n \geq 1)$, if there is another universal R-matrix of $\mathbb{C}[D_{m,n,m-1}]$ except for R_d $(d = 0, 1, \cdots, m-1)$, then $m = 4$ and n is even. In this case, there are exactly 4 such universal R-matrices, and they are given by

$$\tilde{R}_{a,\mu} = \frac{1}{4} \sum_{\alpha,\beta,i,j=0,1} a^{\alpha\beta}(-1)^{\alpha\beta\mu+j\alpha+i\beta} t^\alpha s^{2i+\alpha\mu} \otimes t^\beta s^{2j+\beta\mu},$$

where $a^2 = (-1)^{\frac{n}{2}}$ and $\mu = 0, 1$.

(ii) For the group $D_{4p,4p,2p-1}(p \geq 3)$, there are exactly $4p$ universal R-matrices of $\mathbb{C}[D_{4p,4p,2p-1}]$ except for R_d $(d = 0, 1, \cdots, m-1)$, and they are given by

$$\tilde{R}_c = \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta=0,1} a^{\alpha\beta}(-1)^{\beta\gamma+\alpha\delta} \zeta^{-2c\alpha\beta} t^\alpha s^{c\alpha+2p\gamma} \otimes t^\beta s^{c\beta+2p\delta},$$

where $c = 0, 1, \cdots, 2p-1$, and ζ is a primitive $4p$-th root of unity, and $a^2 = (-1)^c\zeta^{4c}$.

(iii) For the group $D_{4p,4p,2p+1}$ ($p \geq 3$), there are exactly $4p$ universal R-matrices of $\mathbb{C}[D_{4p,4p,2p-1}]$ except for R_d ($d = 0, 1, \cdots, m-1$), and they are given by

$$\tilde{R}_c = \frac{1}{16p^2} \sum_{\substack{\alpha,\beta=0,1 \\ i,j=0,1,\cdots,4p-1}} a^{\alpha\beta}\zeta^{j\alpha+i\beta} \sum_{k,l=1,\cdots,2p} \zeta^{-2(ki+lj)+2c(2kl-k\alpha-l\beta)} t^\alpha s^i \otimes t^\beta s^j,$$

where $c = 0, 1, \cdots, 2p-1$, and ζ is a primitive $4p$-th root of unity, and $a^2 = (-1)^c\zeta^{2c}$.

The above results were showed in the article "二面体群の普遍R行列と結び目の不変量 (Universal R-matrices of dihedral groups and knot invariants)", in Japanese, which published in the proceedings of the 15th symposium on Algebraic Combinatorics, held in Kanazawa University in 1998, p.132–145. Note that the formula of $\tilde{R}_{a,\mu}$ in the above theorem is slightly different from that in Theorem 1.2. The difference arises from the relation $t^{-1}st = s^{-1}$ in $D_{4,n,3}$. (Compare it with the relation $tst^{-1} = s^{-1}$ in $D_{4,n}$.) In deed, the universal R-matrix $\tilde{R}_{a,\mu}$ in Theorem 1.2 corresponds to $\tilde{R}_{-a,\mu}$ in the above theorem by replacing t by t^{-1}.

Department of Mathematics, Faculty of Engineering Science
Kansai University
3-3-35 Yamate-cho, Suita-shi, Osaka 564-8680, Japan