A Small Incongruence Leads to No Information Transmission

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Abstract

This paper analyzes a model of information transmission by cheap talk in which the state of the world and the decision variable are located on a circle. Then the result is that there are only uninformative equilibria even if the incongruence between agents' objectives is small. This is a strikingly different result from one in Crawford and Sobel [2] in which the state and decision are lying on a line.

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1 Introduction

The seminal paper by Crawford and Sobel [2] (hereafter we call CS) is still very influential to the current research on information transmission by cheap talk signals. They analyzed a model in which the state of the world and the relevant decision are modeled as points on an interval in the real line, and showed that if the incongruence between the sender's objective and the receiver's objective is sufficiently small, there exists a perfect Bayesian equilibrium in which some noisy but informative signals are transmitted. We modify the model so that the state of the world and the decision are points on a circle. Then the result is shown to be quite different from one in CS environment. Our main result is that there are only uninformative equilibria under the existence of an incongruence even if it is so small.

To be more precise, our model is the following: In the beginning the state of the world, which is described as a point on a circle, is randomly determined according to the uniform distribution. The realization of the state is observed only by one player, called Sender (S). S sends a costless message to the other player, called Receiver (R), who is uninformed of the state. Based only on the received message, R chooses an action which is described as a point on the same circle as the state space. We restrict our attention to the situation in which R chooses a pure strategy. The state of the world represents the most preferred action for R. R's objective is to minimize the

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square of the distance between the true state and the action. The most preferred action for S is clockwise biased from the state of the world by length b. S's objective is to minimize the square of the distance between his ideal point and the action chosen by R. The positive parameter b is the measure of the incongruence between S's and R's preferences. This is the same model as CS's except that in our model the state and action space is a circle, instead of a line.[1]

Our main result is that in any perfect Bayesian equilibrium R always chooses the same action on the equilibrium path, however small b is. Such an equilibrium is uninformative in the sense that R's action is independent of S's observation and the information transmission never improves the players' ex ante expected payoffs. This result is strikingly different from CS's.

In the literature on location choice models (e.g., Eaton and Lipsey [3]) it is well known that the results in the linear city are quite different from those in the circular city, and such a difference has been attributed to the fact that the former has a "center" and "edges," while the latter does not. This fact also applies to the difference between our model and CS. It is very crucial whether the state and action space has a center and edges or not, which has newly found out.[2]

It is easily verified that if $b = 0$, we have a perfectly informative equilibrium in our model. This implies that in our model, there is some discontinuity of the set of equilibria at $b = 0$. This is also a novel feature we have not observed in CS.

The organization of the paper is as follows: In Section 2 we describe our model and equilibrium concept, and in Section 3 we prove our main theorem.

2 Model

There are two players in the model, a Sender (S) and a Receiver (R); only S has private information. S observes the value of a random variable, m, which is uniformly distributed on $[0, 1)$. The density of m is denoted by $f(m)$. Obviously, $f(m) = 1$ for all $m \in [0, 1)$. S has a von Neumann-Morgenstern utility function $U^S(y, m, b)$, where $y \in [0, 1)$ is the action taken by R upon receiving S's signal and b is a scalar parameter which represents the degree of incongruence between the two players' preferences. R's von Neumann-Morgenstern utility function is denoted $U^R(y, m)$. We consider the utility functions of the following form:

$$\begin{aligned} U^S(y, m, b) &= -\left(d\left(\|m + b\|, y\right)\right)^2, \\ U^R(y, m) &= -\left(d(m, y)\right)^2, \end{aligned} \tag{1}$$

where

$$d(x, x') = \min\left\{|x - x'|, 1 - |x - x'|\right\},$$

[1] In CS, the case of uniform distribution and quadratic utilities is extensively analyzed in Section 4.

[2] Recent literature, e.g., Battaglini [1], analyzes an extension of CS model where the state space and the set of available actions are multi-dimensional. This framework still has a center and edges as in CS.

and

$$\|x\| = \begin{cases} x+1 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ x-1 & \text{if } x \geq 1 \end{cases} .$$

In other words, we identify $[0, 1)$ with a circle with circumference 1. The parameter b is the distance between the ideal actions for S and R. We assume $0 < b \leq 1/2$.

We follow the definition of an equilibrium by CS. In CS, strong concavity of R's utility function implies uniqueness of his best response and thus he plays a pure strategy in any equilibrium. The definition of an equilibrium by CS rules out the possibility that R may play a mixed strategy. In our analysis, since R's utility is not concave with respect to the domain $[0, 1)$, we should consider the possibility that a mixed strategy is played by R in an equilibrium. However, in this paper, we limit our attention to equilibria in which R plays a pure strategy and use the same definition of an equilibrium as CS. To consider how our argument changes when we allow R to play mixed strategies is left as future research.

An equilibrium consists of a signaling rule for S, denoted $q(n|m)$, and an action rule for R, denoted $y(n)$, such that:

(1) for each $m \in [0, 1)$, $\int_N q(n|m)dn = 1$, where the Borel set N is the set of feasible signals, and if n^* is in the support of $q(\cdot|m)$, then n^* solves $\max_{n \in N} U^S(y(n), m, b)$; and

(2) for each $n \in N$, $y(n)$ solves $\max_y \int_0^1 U^R(y, m)p(m|n)dm$,
where $p(m|n) \equiv q(n|m)f(m) \Big/ \int_0^1 q(n|t)f(t)dt$.

3 Equilibrium

We call an equilibrium *uninformative* if only one action is chosen on the equilibrium path. Such an equilibrium is uninformative in the sense that R's action is independent of S's observation and the information transmission never improves the players' ex ante expected payoffs. Note that there always exists a trivial uninformative equilibrium, i.e., any S-type always sends the same message $n^* \in N$, $p(\cdot|n)$ is the density function of the uniform distribution on $[0, 1)$ for any $n \in N$, and R always chooses the same action y^*. Our main theorem tells that there are only uninformative equilibria.

Theorem 1 Any equilibrium is uninformative.

Proof:
For each action $\overline{y} \in [0, 1)$, let $N(\overline{y}) \equiv \{n : y(n) = \overline{y}\}$. We say that an action $\overline{y}$ is *induced* by an S-type $\overline{m}$ if $\int_{N(\overline{y})} q(n \mid \overline{m}) > 0$. Let Y be the set of all actions induced by some S-type. The proof consists of two steps. Lemma 1 shows that the set of actions induced in equilibrium is finite. Lemma 2 through 4 show that if finite but multiple actions are induced in equilibrium we reach a contradiction. Then we can conclude only one action is induced in any equilibrium.

Lemma 1 The set of actions induced in equilibrium Y is finite.

Proof:
Suppose not. Then for any $\epsilon > 0$, there exist three actions induced in equilibrium y, y' and y'', with $y' < y < y''$, $y - y' < \epsilon$ and $y'' - y < \epsilon$. Without loss of generality (or by selecting an appropriate point of the circle as the origin), let $y = \frac{1}{4} + b$. An S-type $m \in \left[0, \frac{1}{4} - \epsilon\right) \cup \left[\frac{3}{4}, 1\right)$ strictly prefers y' to y. An S-type $m \in \left(\frac{1}{4} + \epsilon, \frac{3}{4}\right)$ strictly prefers y'' to y. Thus, y is induced only by S-types $m \in \left[\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon\right]$. Therefore, for any message n such that $y(n) = y$, $p(m|n)$ puts positive weight only on the interval $\left[\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon\right]$. Then, for ϵ sufficiently small relative to b, we have

$$\int_0^1 U^R\left(\frac{1}{4}, m\right) p(m|n)dm > \int_0^1 U^R(y, m)p(m|n)dm,$$

violating R's incentive condition. Contradiction. ∎

Suppose, to the contrary to Theorem, that Y contains $K \geq 2$ actions throughout the proof. Let $Y = \{y_1, \ldots, y_K\}$.

Lemma 2 For all $y_i \in Y$, there exists $M_i \subset [0, 1)$ which is an closed interval on the circle, i.e., either $M_i = [\underline{m}_i, \overline{m}_i]$ or $M_i = [0, \overline{m}_i] \cup [\underline{m}_i, 1)$, such that

$$U^S(y_i, m, b) \geq U^S(y', m, b) \quad \forall y' \in Y \setminus \{y_i\}$$

if and only if $m \in M_i$.

Proof:
Without loss of generality, consider the case of $y_i = b$. For each $y' \in Y \setminus \{y_i\}$, let

$$\overline{\mu}(y') = \frac{\|y' - b\|}{2} \quad \text{and} \quad \underline{\mu}(y') = \frac{1}{2} + \frac{\|y' - b\|}{2}.$$

From S's utility function, it can be easily verified that for all $y' \in Y \setminus \{y_i\}$,

$$U^S(y_i, m', b) \geq U^S(y', m', b) \quad \text{iff } m' \in \left[0, \overline{\mu}(y')\right] \cup \left[\underline{\mu}(y'), 1\right).$$

Therefore, $U^S(y_i, m', b) \geq U^S(y', m', b)$ for all $y' \in Y \setminus \{y_i\}$ if and only if

$$m' \in \cap_{y' \in Y \setminus \{y_i\}} \left(\left[0, \overline{\mu}(y')\right] \cup \left[\underline{\mu}(y'), 1\right)\right). \tag{2}$$

Notice $\overline{\mu}(y') < 1/2$ for all y' and $\underline{\mu}(y') > 1/2$ for all y'. Thus (2) is equivalent to

$$m' \in \left(\cap_{y' \in Y \setminus \{y_i\}} \left[0, \overline{\mu}(y')\right]\right) \cup \left(\cap_{y' \in Y \setminus \{y_i\}} \left[\underline{\mu}(y'), 1\right)\right). \tag{3}$$

Define $\underline{m}_i$ and $\overline{m}_i$ by $\underline{m}_i = \max_{y' \in Y \setminus \{y_i\}} \underline{\mu}(y')$ and $\overline{m}_i = \min_{y' \in Y \setminus \{y_i\}} \overline{\mu}(y_{i+1})$. Then, (3) is equivalent to

$$m' \in [0, \overline{m}_i] \cup [\underline{m}_i, 1).$$

∎

From S's utility function, for $y_i, y_j \in Y$ with $i \neq j$, there are only two values of m for which $U^S(y_i, m, b) = U^S(y_j, m, b)$ holds. Therefore, M_i and M_j intersect at at most two points. Thus, M_i may intersect with M_j, $j \neq i$ only at $\underline{m}_i$ and $\overline{m}_i$. Also, any $m \in [0, 1)$ must be included in some $i \in \{1, \ldots, K\}$, because there is at least one $y_i \in Y$ that maximizes $U^S(y, m, b)$. Then it follows from Lemma 1 and 2 that there is a finite partition of $[0, 1)$ such that the closed hull of each element is exactly corresponding to one of $\{M_i\}_{i \in \{1,\ldots,K\}}$.

Let $\text{int}(M_i)$ be the interior of M_i. From the above argument, if $m \in \text{int}(M_i)$, then $m \notin M_j$, $j \neq i$, and thus

$$U^S(y_i, m, b) > U^S(y_j, m, b).$$

Therefore, in equilibrium any S-type $m \in \text{int}(M_i)$ chooses a message $n \in N(y_i)$. Formally, for all $m \in \text{int}(M_i)$,

$$\int_{N(y_i)} q(n|m) dn = 1. \tag{4}$$

Lemma 3 For any $y_i \in Y$,

(1) if $M_i = [\underline{m}_i, \overline{m}_i]$, then $y_i = \frac{\underline{m}_i + \overline{m}_i}{2}$,

(2) if $M_i = [0, \overline{m}_i] \cup [\underline{m}_i, 1)$, then $y_i = \left\| \frac{\underline{m}_i + (1 + \overline{m}_i)}{2} \right\|$.

Proof:
Without loss of generality, we prove only the first case. For any $n \in N(y_i)$, R's expected payoff when he chooses action y conditionally on observing n is

$$\int_0^1 U^R(y, m)\, p(m|n)\, dm. \tag{5}$$

Since $f(m) = 1$ for all m, for each fixed $n \in N$, $p(m|n)$ is proportional to $q(n|m)$. Therefore, (5) is proportional to

$$\int_0^1 U^R(y, m)\, q(n|m)\, dm. \tag{6}$$

Furthermore, since $n \in N(y_i)$, we have $q(n|m) = 0$ for all $m \notin M_i$. Thus, (6) is equal to

$$\int_{\underline{m}_i}^{\overline{m}_i} U^R(y, m)\, q(n|m)\, dm.$$

Since y_i must maximize it, we have

$$\int_{\underline{m}_i}^{\overline{m}_i} U^R(y_i, m)\, q(n|m)\, dm \geq \int_{\underline{m}_i}^{\overline{m}_i} U^R(y', m)\, q(n|m)\, dm, \quad \forall y'.$$

Since the above inequality holds for any $n \in N(y_i)$, we obtain

$$\int_{\underline{m}_i}^{\overline{m}_i} \int_{N(y_i)} U^R(y_i, m)\, q(n|m)\, dndm \geq \int_{\underline{m}_i}^{\overline{m}_i} \int_{N(y_i)} U^R(y', m)\, q(n|m)\, dndm, \quad \forall y',$$

which is, by (4), equivalent to

$$\int_{\underline{m}_i}^{\overline{m}_i} U^R(y_i, m)\, dm \geq \int_{\underline{m}_i}^{\overline{m}_i} U^R(y', m) dm, \quad \forall y'.$$

Therefore, from R's utility function,

$$y_i = \frac{\underline{m}_i + \overline{m}_i}{2}$$

must hold. ∎

For each $i \in \{1, \ldots, K\}$, let $l(M_i)$ be the length of M_i. Formally,

$$l(M_i) = \begin{cases} \overline{m}_i - \underline{m}_i & \text{if } M_i = [\underline{m}_i, \overline{m}_i], \\ 1 + \overline{m}_i - \underline{m}_i & \text{if } M_i = [0, \overline{m}_i] \cup [\underline{m}_i, 1). \end{cases}$$

Lemma 4 For any $y_i, y_j \in Y$ with $i \neq j$, if $\overline{m}_i = \underline{m}_j$, then $l(M_j) = l(M_i) + 4b$.

Proof:
Without loss of generality, consider the case where $y_i = 0$ and $y_j > 0$. Let $m^* \equiv \overline{m}_i = \underline{m}_j$. From Lemma 2, m^* must satisfy

$$U^S(y_i, m^*, b) = U^S(y_j, m^*, b),$$
$$\frac{\partial}{\partial m} U^S(y_i, m^*, b) < \frac{\partial}{\partial m} U^S(y_j, m^*, b).$$

Therefore,

$$m^* = \left\| \frac{y_j}{2} - b \right\|.$$

Since Lemma 3 implies $y_i \in M_i$ and $y_j \in M_j$, we must have $0 < m^* < y_j$ and thus

$$m^* = \frac{y_j}{2} - b.$$

Now, from Lemma 3,

$$l(M_i) = 2m^* = y_j - 2b$$

and

$$l(M_j) = 2(y_j - m^*) = y_j + 2b,$$

which proves the Lemma. ∎

It is obvious that there is no consistent $\{M_i\}_{i \in \{1, \ldots, K\}}$ for any $K \geq 2$. Contradiction. Then it follows that any S-type induces one common action. ∎

References

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