Hysteresis in Dynamic Geneal Equilibrium Models with Cash-in-Advance Constraints

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Very Preliminary

Results

- We present dynamic Walrasian market models with cash-in-advance constraints
- **Indederminacy Result:** there is a continuum of equilibria with deterministic cycles having same periodicity and different amplitudes
- **Hysteresis result:** different initial money holdings distributions lead to different limit cycles

Roadmap

- Example model
  - Equilibria with 2-period cycles
  - Equilibria with **T**-period cycles
  - Dynamic equilibria leading to the limit cycle
- Extensions of example model
- (General model)

Example Model

- Time is discrete $\mathbf{t = 1, 2, \ldots}$
- A continuum of infinitely lived agents with measure $\mathbf{1}$
- No double coincidence of wants
- $\mathcal{T}$ types of agents with equal fractions
- $\mathcal{T}$ types of goods: non-durable and divisible
- type τ agent can produce one unit of type $\tau + \mathbf{1}$ good with cost $\mathbf{c > 0}$
- type τ agent obtains $\mathbf{U(q) = aq}$ only if she consumes $\mathbf{q}$ amount of type τ good

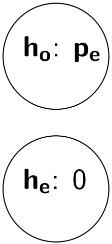
Example Model (cont'd)

- Fiat money: durable and divisible
- $\mathbf{M} > \mathbf{0}$: nominal stock of fiat money (fixed)
- $\delta \in (\mathbf{0}, \mathbf{1})$: discount factor
- Cash-in-advance constraint:
- Participation constraint:
  In the begining of each time, each agent chooses whether to be a seller, to be a buyer, or to do nothing

Equilibria with 2-period cycles

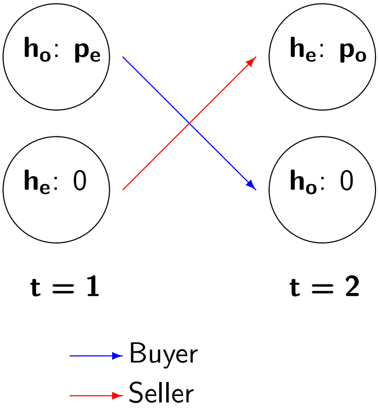
- seller $\longrightarrow$ buyer $\longrightarrow$ seller ...
- $\mathbf{p_i}$: price at time $\mathbf{i}$ ($\mathbf{i} = \underline{\mathbf{o}}\mathbf{dd}, \underline{\mathbf{e}}\mathbf{ven}$)
- $\mathbf{h_i}$: measure of agents with $\mathbf{p_j}$ amount of money at the begining of time $\mathbf{i}$ ($\mathbf{j} \neq \mathbf{i}$)

Transition of Money Holdings

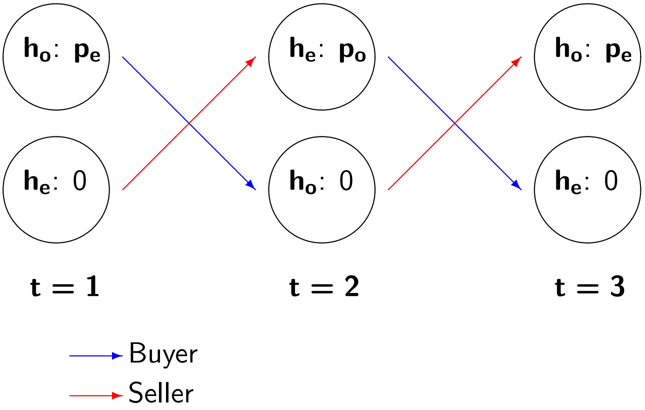


$$t = 1$$

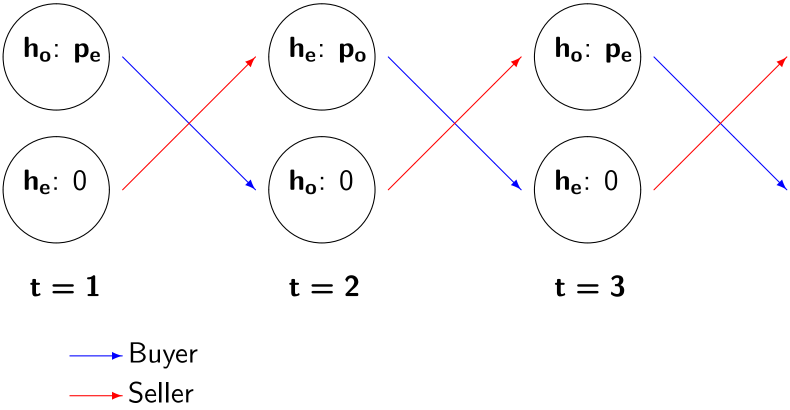
Transition of Money Holdings



Transition of Money Holdings



Transition of Money Holdings



Equilibrium Conditions (except Optimality Condition)

- Condition for stationary cycle

$$\mathbf{h_o + h_e = 1}$$

- Condition for constant stock of fiat money

$$\mathbf{M = h_o p_e}$$
$$\mathbf{M = h_e p_o}$$

- Condition for market clearing

$$\mathbf{h_o p_e = h_e p_o}$$
$$\mathbf{h_e p_o = h_o p_e}$$

Construction of Equilibria with 2-Period Cycles

- We construct a time-invariant equilibrium with 2 states
- $\mathbf{h_o = h_e = \frac{1}{2}}$
- $\mathbf{p = p_o = p_e = 2M}$
- We transform it into a continuum of equilibria with 2-period cycle by perturbing $\mathbf{(h_o, h_e)}$
- $\mathbf{h_o = \frac{1}{2} - \epsilon}$, $\mathbf{h_e = \frac{1}{2} + \epsilon}$
- $\mathbf{p_o = \frac{M}{1/2+\epsilon}}$, $\mathbf{p_e = \frac{M}{1/2-\epsilon}}$

Strategy of Time Invariant Equilibrium

- An agent with $\eta \in [0, \bar{\eta}]$ chooses to be a seller
- An agent with $\eta \in (\bar{\eta}, \infty)$ chooses to be a buyer and spends all her money
- We choose $\bar{\eta}$ so that
  - an agent is indifferent between being a seller and a buyer at $\bar{\eta}$
  - $\bar{\eta} \in (0, \mathbf{p})$

Optimality Condition

- If $-1 + \delta + \delta^2 < \mathbf{c}/\mathbf{a} < \delta$, the optimality condition holds in strict inequalities (except at $\bar{\eta}$)
- The optimality condition can hold even under small perturbation of $\mathbf{h}$ by appropriately adjusting $\bar{\eta}$

Equilibria with $\mathbf{T}$-Period Cycle

- seller: $(\mathbf{T}-\mathbf{1})$ times $\longrightarrow$ buyer $\longrightarrow$ seller: $(\mathbf{T}-\mathbf{1})$ times $\longrightarrow$ $\cdots$
- $\mathbf{p_i}$: price at time $\mathbf{Tn}+\mathbf{i}$ ($\mathbf{i}=\mathbf{0},\mathbf{1},\ldots,\mathbf{T}-\mathbf{1}$)
- $\mathbf{h_i}$: measure of agents with no money at time $\mathbf{Tn}-\mathbf{i}$

Equilibrium Conditions (except Optimality Condition)

- Endogenous variables: **2T**
- Condition for stationary cycle: **1** equation
- Condition for constant stock of fiat money: **T** equations
- Condition for market clearing: Redundant
- $\Rightarrow$ $\mathbf{(T-1)}$ degrees of freedom in determination of endogenous variables

Construction of Equilibria with $\mathbf{T}$-Cycles

- We construct a time-invariant equilibria with $\mathbf{T}$ states
- $\mathbf{h_0 = h_1 = \cdots = h_{T-1} = \frac{1}{T}}$
- $\mathbf{p_0 = p_1 = \cdots = p_{T-1} = \frac{2}{T-1}M}$
- We transform it into a continuum of equilibria with $\mathbf{T}$-period cycles by perturbing $\mathbf{h = (h_0, h_1, \ldots, h_{T-1})}$

Strategy of Time Invariant Equilibrium

- An agent with $\eta \in [0, \bar{\eta}]$ sells her production good
- An agent with $\eta \in (\bar{\eta}, \infty)$ spends all her money as a buyer
- We choose $\bar{\eta}$ so that
  - an agent is indifferent between being a seller and a buyer at $\bar{\eta}$
  - $\bar{\eta} \in ((\mathbf{T}-2)\mathbf{p}, (\mathbf{T}-1)\mathbf{p})$

Optimality Condition

- Suppose $\delta - (\mathbf{T}-1)(1-\delta^2) < \mathbf{c}/\mathbf{a} < \delta - (\mathbf{T}-2)(1-\delta^2)$
- Optimality condition holds in strict inequalities (except at $\bar{\eta}$)
- Small perturbation of $\mathbf{h}$ does not violate the optimality by setting $\bar{\eta}$ appropriately

Summary for Indeterminacy Result

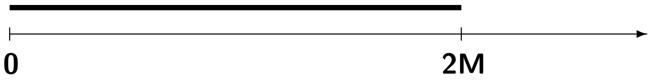
- Equilibrium condition:
  $\delta - (T-1)(1-\delta^2) < c/a < \delta - (T-2)(1-\delta^2)$
- Conditions of different periodicity have no intersection
- $\Leftrightarrow 1 + \frac{\delta - c/a}{1-\delta^2} < T < 2 + \frac{\delta - c/a}{1-\delta^2}$
- $\delta \to 1 \Rightarrow T \to \infty$
- $\because$ average consumption: $\frac{T-1}{T}$

Dynamic Equilibria Leading to 2-Period Cycle

- Suppose that the initial money holdings distribution is uniform over $\mathbf{[0, 2M]}$

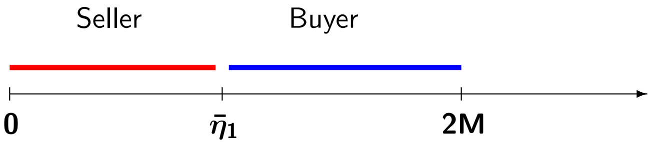
Transition of Money Holdings

$\mathbf{t = 1}$ (Transition Stage)



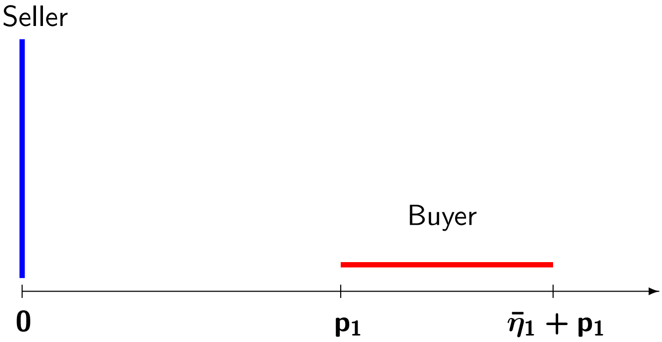
Transition of Money Holdings

$\mathbf{t = 1}$ (Transition Stage)



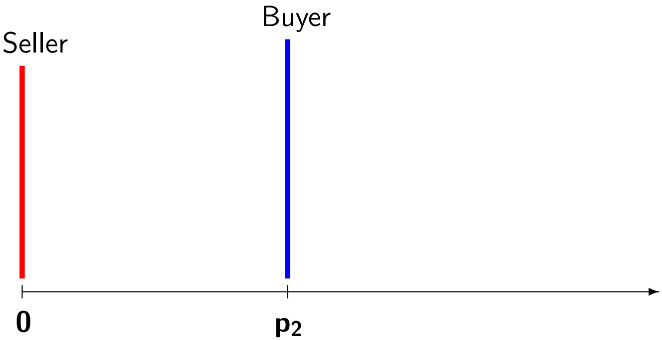
Transition of Money Holdings

$\mathbf{t = 2}$ (Transition Stage)



Transition of Money Holdings

$\mathbf{t = 3}$ (Cycle Stage)



Equilibrium Condition for Dynamic Equilibria Leading to 2-Period Cycle

- Market clearing condition at $t = 1$ determines $\mathbf{p_1}$
- Market clearing condition at $t = 2$ determines $\mathbf{p_2}$
- $\mathbf{h_o} = \#$ Buyers at $\mathbf{t = 1}$
- If there exists a dynamic equilibrium with the above features, it is necessarily unique
- Different $\mathbf{c/a}$ brings about different $\mathbf{h_o}$

Redistribution Policy on the Initial Distribution

- Initial density:

$$f_1(\eta) = \begin{cases} \frac{1}{2M} & \eta \in (2M\epsilon, 2M(1-\epsilon)] \\ \frac{1}{M} & \eta \in [2M(1-\epsilon), 2M] \\ 0 & \text{otherwise} \end{cases}$$

- Different ϵ leads to different equilibrium with 2-period cycles
- Such a policy has a permanent effect (Hyseresis)

Dynamic Equilibria Leading to 3-Period Cycle

- Price converges to unique limit cycle with 3-period, but not in finite time
- Convergence of 2-Period cycle:
  $\hat{\mathbf{p}}_2 \mathbf{h}_o = \mathbf{M}$ must hold at the beginning of $\mathbf{t = 3}$
  $\Rightarrow \hat{\mathbf{p}}_2 = \mathbf{p}_e$ must hold
  $\Rightarrow$ price converges at $\mathbf{t = 3}$
- This property cannot be extended to 3-period cycle
- $\because \mathbf{h}_0 \hat{\mathbf{p}}_3 + \mathbf{h}_1(\hat{\mathbf{p}}_2 + \hat{\mathbf{p}}_3) = \mathbf{M}$ does not uniquely determine $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{p}}_3$

Extensions

- Non-linear utility function
- Convex cost function
- Small stochastic preference shock
- No participation constraint with durable goods

Small Stochastic Preference Shock

- $\mathbf{U(q) = \theta q}$
- $\theta \sim \mathbf{U[a - \Theta, a + \Theta]}$ $(\Theta > 0)$
- Equilibium conditions for equilibria with **2**-period cycles:
  - $-1 + \delta + \delta^2 < \mathbf{c/a} < \delta$
  - $\Theta < \dfrac{\mathbf{a}(1 - \delta - \delta^2) + \mathbf{c}}{1 + \delta}$

Contrast between Lucas (1980)'s and Our Results

- Lucas (1980):
  - Assumption: $\lim_{\theta \to \bar{\theta}} \frac{\partial U(q, \theta)}{\partial q} = \infty$
  - Theorem: Limit distribution is unique
- Stochastic porcess of money holdings distribubtion...
  - can be **periodic** under small stochastic shock, but
  - must be **ergodic** under large stochastic schock

General Model

Under construction!