

Supplement to “Stationary Monetary Equilibria with Strictly Increasing Value Functions and Non-Discrete Money Holdings Distributions: An Indeterminacy Result”

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This note is a supplemental material to Kamiya and Shimizu [1] (hereafter, KS). We prove Theorems 2 and 3 in KS.

1 Proof of Theorem 2

Theorem 2 Suppose agents can hold any amount of money, i.e., $B = \infty$. Suppose $\frac{3}{2} < d \leq 3$. Let $\underline{\beta} = \frac{3k}{3(k-1)+2d}$. Then, for any given $\beta \in (\underline{\beta}, 1)$, there exists a continuum of stationary equilibria in which (i) the value functions are continuous, strictly increasing, and concave, and (ii) the money holdings distributions have a full support in some closed interval with a nonempty interior.

Proof:

(I) We extend the strategy and money holdings distribution constructed in the proof of Theorem 1 of KS to the environment without an upper bound of individual money holdings. In other words, for some $p > 0$,

- an agent without money always chooses to be a seller and an agent with money holding $\eta > 0$ always chooses to be a buyer,
- a seller always offers (p, \bar{q}) ,

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- a buyer with money holding $\eta > 0$ consumes the following amount of her consumption good: there exists a $\underline{p}(\eta) \geq p$ such that, for given (p_s, q_s) ,

$$q_b(\eta, p_s, q_s) = \begin{cases} \min\{\eta/p_s, q_s\} & \text{if } p_s \leq \underline{p}(\eta), \\ 0 & \text{if } p_s > \underline{p}(\eta), \end{cases} \quad (1)$$

- for some λ and σ , f is expressed by

$$f(\eta) = \begin{cases} 2\lambda\eta + \sigma, & \text{for } \eta \in (0, p\bar{q}], \\ 0, & \text{for } \eta \in (p\bar{q}, \infty]. \end{cases} \quad (2)$$

Note that p , $\underline{p}(\eta)$, λ , and σ will be determined as functions of m_0 later.

(II) Next, we obtain a candidate for a value function $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ consistent with the above strategy. From the above strategy, $V(\eta)$ for $\eta \in (0, \infty)$ can be written as a function of $V(0)$ as follows. First, for $\eta \in (0, p\bar{q}]$,

$$V(\eta) = \frac{m_0}{k} \left(a \frac{\eta}{p} + \beta V(0) \right) + \left(1 - \frac{m_0}{k} \right) \beta V(\eta)$$

holds. Thus

$$V(\eta) = A(m_0) \left(a \frac{\eta}{p} + \beta V(0) \right), \quad \text{for } \eta \in (0, p\bar{q}], \quad (3)$$

where $A(m_0) = \frac{m_0}{k - (k - m_0)\beta}$. Note that $A(m_0) < 1$. Similarly, $V(\eta)$ for $\eta \in (p\bar{q}, \infty)$ is written as:

$$V(\eta) = A(m_0) (a\bar{q} + \beta V(\eta - p\bar{q})), \quad \text{for } \eta \in (p\bar{q}, \infty). \quad (4)$$

Next, since an agent without money always chooses to be a seller, then $V(0)$ is determined by

$$V(0) = \frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta V(\eta) \frac{f(\eta)}{1 - m_0} d\eta \right] + \left(1 - \frac{1 - m_0}{k} \right) \beta V(0). \quad (5)$$

(III) Below, we focus on equilibria with $V(0) = 0$ and obtain (p, λ, σ) as functions of m_0 . First, we decompose $\eta \geq 0$ into an multiple of $p\bar{q}$ and a residual; that is, $\eta = np\bar{q} + \iota$, where

n is a nonnegative integer and ι is a nonnegative real number less than $p\bar{q}$. Then, by (3) and (4),

$$V(np\bar{q} + \iota) = \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^n \left[\bar{q} - (1 - \beta A(m_0)) \frac{\iota}{p} \right] \right\} \quad (6)$$

holds. On the other hand, by (2) and (5),

$$(1 - m_0)\bar{c} = \frac{a\beta A(m_0)}{p} \int_{(0, p\bar{q}]} \eta f(\eta) d\eta = a\beta A(m_0) \left(\frac{2}{3} \lambda p^2 \bar{q}^3 + \frac{1}{2} \sigma p \bar{q}^2 \right) \quad (7)$$

holds.

Below, we obtain (p, λ, σ) as functions of m_0 . First, $1 - m_0 = \int_{(0, \infty)} f d\eta$ can be written as follows:

$$1 - m_0 = \int_{(0, p\bar{q}]} f d\eta = \lambda p^2 \bar{q}^2 + \sigma p \bar{q}. \quad (8)$$

Since the total amount of money the agents have is equal to M , the following equation must be satisfied:

$$M = \int_{(0, p\bar{q}]} \eta f d\eta = \frac{2}{3} \lambda p^3 \bar{q}^3 + \frac{1}{2} \sigma p^2 \bar{q}^2. \quad (9)$$

By (7), (8), (9), and $d = \frac{a\bar{q}}{c}$, we obtain

$$p = \frac{Ma\beta A(m_0)}{(1 - m_0)\bar{c}}, \quad (10)$$

$$\lambda = \frac{3(1 - m_0)^3 (2 - \beta d A(m_0))}{M^2 \beta^3 d^3 (A(m_0))^3}, \quad (11)$$

$$\sigma = \frac{2(1 - m_0)^2 (-3 + 2\beta d A(m_0))}{M \beta^2 d^2 (A(m_0))^2}. \quad (12)$$

(IV) Next, we check the optimality of the specified strategy.

(i) The optimality of the strategy of an agent with money holding $\eta > 0$:

First, we show that there exists a $\underline{p}(\eta) \geq p$ in (1). If $\eta \in (0, p\bar{q}]$, then by (6),

$$aq + \beta V(\eta - p_s q) = aq \left(1 - \beta A(m_0) \frac{p_s}{p} \right) + a\beta A(m_0) \frac{\eta}{p}$$

holds. Thus if

$$1 - \beta A(m_0) \frac{p_s}{p} \geq 0$$

holds, then she clearly chooses the maximum amount she can buy, and otherwise she chooses $q_b = 0$. Note that $1 - \beta A(m_0) \geq 0$ holds, since $\beta A(m_0) < 1$. Let

$$\underline{p}(\eta) = \frac{p}{\beta A(m_0)}, \quad \text{for } \eta \in (0, p\bar{q}]. \quad (13)$$

Then, (1) is optimal for $\eta \in (0, p\bar{q}]$. Moreover, $\underline{p}(\eta) \geq p$ clearly holds. Similar arguments apply to the case of $\eta \in (p\bar{q}, \infty)$.

Next, we check an incentive for an agent with $\eta > 0$ to become a buyer instead of becoming a seller and offering (p', q') . By (1) and (13), for any $p' > \frac{p}{\beta A(m_0)}$, no buyer accepts such an offer on the equilibrium, and then the value is the same as that of an offer $(p'', 0)$, where $p'' \leq \frac{p}{\beta A(m_0)}$. Therefore, we can restrict our attention to (p', q') such that $p' \in \left[0, \frac{p}{\beta A(m_0)}\right]$ and $q' \in [0, \bar{q}]$. By (1), the value of becoming a seller and offering (p', q') is

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta \tilde{V}(\eta, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right] + \left(1 - \frac{1 - m_0}{k} \right) \beta V(\eta),$$

where

$$\tilde{V}(\eta, \eta') = \begin{cases} V(\eta + \eta'), & \text{if } \eta' \leq p'q', \\ V(\eta + p'q'), & \text{if } \eta' > p'q'. \end{cases} \quad (14)$$

On the other hand, when she becomes a buyer, the value is $V(\eta)$. Thus the difference is

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta \left(\tilde{V}(\eta, \eta') - V(\eta) \right) \frac{f(\eta')}{1 - m_0} d\eta' \right] - (1 - \beta)V(\eta). \quad (15)$$

Below, we show

$$V(\eta + \eta') - V(\eta) \leq aA(m_0) \frac{\eta'}{p}, \quad \text{for } \eta' \in (0, p\bar{q}]. \quad (16)$$

First, there exists a unique nonnegative integer n such that $np\bar{q} \leq \eta < (n+1)p\bar{q}$. There are two cases: (a) $\eta + \eta' < (n+1)p\bar{q}$ and (b) $\eta + \eta' \geq (n+1)p\bar{q}$. In case (a), by (6) and $\beta A(m_0) < 1$,

$$\begin{aligned} V(\eta + \eta') - V(\eta) &= \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^n \left[\bar{q} - (1 - \beta A(m_0)) \frac{\eta + \eta' - np\bar{q}}{p} \right] \right\} \\ &\quad - \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^n \left[\bar{q} - (1 - \beta A(m_0)) \frac{\eta - np\bar{q}}{p} \right] \right\} \\ &\leq aA(m_0) (\beta A(m_0))^n \frac{\eta'}{p} \\ &\leq aA(m_0) \frac{\eta'}{p}. \end{aligned}$$

In case (b), by (6) and $\beta A(m_0) < 1$,

$$\begin{aligned}
V(\eta + \eta') - V(\eta) &= \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^{n+1} \left[\bar{q} - (1 - \beta A(m_0)) \frac{\eta + \eta' - (n+1)p\bar{q}}{p} \right] \right\} \\
&\quad - \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^n \left[\bar{q} - (1 - \beta A(m_0)) \frac{\eta - np\bar{q}}{p} \right] \right\} \\
&= aA(m_0) (\beta A(m_0))^n \left[(1 - \beta A(m_0)) (n+1)\bar{q} + \beta A(m_0) \frac{\eta'}{p} - (1 - \beta A(m_0)) \frac{\eta}{p} \right] \\
&\leq aA(m_0) (\beta A(m_0))^n \frac{\eta'}{p} \\
&\leq aA(m_0) \frac{\eta'}{p}.
\end{aligned}$$

The fourth line is obtained by $\eta \geq (n+1)p\bar{q} - \eta'$. This completes the proof of (16).

(6), (14), and (16) imply

$$\tilde{V}(\eta, \eta') - V(\eta) \leq aA(m_0) \frac{\eta'}{p}, \quad \text{for } \eta' \in (0, p\bar{q}].$$

Then, the first term of (15) is less than or equal to

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} a\beta A(m_0) \frac{\eta'}{p} \frac{f(\eta')}{1 - m_0} d\eta' \right].$$

This is equal to zero by the first equality of (7), and thus (15) is non-positive and she becomes a buyer.

(ii) The optimality of the strategy of an agent without money:

By the construction, an agent without money is indifferent between a buyer and a seller. Thus she has an incentive to be a seller. As in the latter part of (i), we restrict our attention to offers (p', q') such that $p' \in \left[0, \frac{p}{\beta A(m_0)}\right]$ and $q' \in [0, \bar{q}]$. By (1) and (6), the value of offering (p', q') is

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta \tilde{V}(0, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right], \quad (17)$$

where \tilde{V} is defined in (14). If $p'q' \geq p\bar{q}$, $\tilde{V}(0, \eta') = V(\eta')$ for any $\eta' \in (0, p\bar{q}]$. Then, (17) is

the same for all $p'q' \geq p\bar{q}$, and therefore the offer (p, \bar{q}) is optimal. If $p'q' \leq p\bar{q}$,

$$\begin{aligned} \int_{(0, p\bar{q}]} \tilde{V}(0, \eta') f(\eta') d\eta' &= \int_{(0, p'q']} V(\eta') f(\eta') d\eta' + \int_{(p'q', p\bar{q}]} V(p\bar{q}) f(\eta') d\eta' \\ &\leq \int_{(0, p'q']} V(\eta') f(\eta') d\eta' + \int_{(p'q', p\bar{q}]} V(\eta') f(\eta') d\eta' \\ &= \int_{(0, p\bar{q}]} \tilde{V}(0, p\bar{q}) f(\eta') d\eta', \end{aligned}$$

where the inequality is obtained by (6) and (14). Then, the offer (p, \bar{q}) is optimal. This completes the proof of (IV).

(V) Finally, we check $f(\eta) \geq 0$ for all $\eta \in (0, p\bar{q}]$. Since f is linear, it suffices to show $f(0) \geq 0$ and $f(p\bar{q}) \geq 0$. By (10), (11), and (12),

$$f(0) = \sigma = \frac{2(1 - m_0)^2(-3 + 2\beta dA(m_0))}{M\beta^2 d^2 (A(m_0))^2}$$

and

$$f(p\bar{q}) = 2\lambda p\bar{q} + \sigma = \frac{2(1 - m_0)^2(3 - \beta dA(m_0))}{M\beta^2 d^2 (A(m_0))^2}$$

hold. A sufficient condition for $f(0) \geq 0$ and $f(p\bar{q}) \geq 0$ is clearly

$$\frac{3}{2} \leq \beta dA(m_0) \leq 3.$$

By the assumption $d \leq 3$, $\beta dA(m_0) \leq 3$ is always satisfied. It is easily verified that $\frac{3}{2} \leq \beta dA(m_0)$ is equivalent to

$$m_0 \geq \frac{3k(1 - \beta)}{\beta(2d - 3)}. \quad (18)$$

Setting $\underline{\beta} = \frac{3k}{3(k-1)+2d}$, we can show that for any $\beta \in (\underline{\beta}, 1)$ there exists a continuum of m_0 satisfying (18) and $m_0 \in (0, 1)$. Indeed, $\underline{\beta} < 1$ follows from the assumption $d > \frac{3}{2}$, and $1 > \frac{3k(1-\beta)}{\beta(2d-3)}$ follows from $\beta > \underline{\beta}$.

The stationary condition is expressed as follows:

$$m_0 \frac{1 - m_0}{k} \int_{(\eta, p\bar{q}]} \frac{f(\eta')}{1 - m_0} d\eta' = \int_{(\eta, p\bar{q}]} f(\eta') \frac{m_0}{k}, \quad \text{for } \eta \in [0, p\bar{q}],$$

where the LHS is the outflow from $[0, \eta]$, while the RHS is the inflow into $[0, \eta]$. Nevertheless, this is automatically satisfied. This concludes the proof. \blacksquare

2 Proof of Theorem 3

Theorem 3 Suppose $\frac{3}{2} < d \leq 3$. Let $\underline{\beta} = \frac{3k}{3(k-1)+2d}$. Then, for any given $\beta \in (\underline{\beta}, 1)$ and $B > \underline{B} = \frac{3M\beta(2d-3)}{2(\beta(2d-3)-3k(1-\beta))}$, there exists a continuum of stationary equilibria satisfying the commodity-money refinement in the sense of Zhou.

Proof:

Throughout the proof, we assume $\varepsilon \geq 0$.

(I) We consider the same strategy and money holdings distribution as in KS: for some $p \in \left(0, \frac{B}{\bar{q}}\right]$,

- an agent without money always chooses to be a seller and an agent with money holding $\eta > 0$ always chooses to be a buyer,
- a seller always offers (p, \bar{q}) ,
- a buyer with money holding $\eta > 0$ consumes the following amount of her consumption good: there exists a $\underline{p}(\eta) \geq p$ such that, for given (p_s, q_s) ,

$$q_b(\eta, p_s, q_s) = \begin{cases} \min\{\eta/p_s, q_s\} & \text{if } p_s \leq \underline{p}(\eta), \\ 0 & \text{if } p_s > \underline{p}(\eta), \end{cases} \quad (19)$$

- for some λ and σ , f is expressed by

$$f(\eta) = \begin{cases} 2\lambda\eta + \sigma, & \text{for } \eta \in (0, p\bar{q}], \\ 0, & \text{for } \eta \in (p\bar{q}, \infty]. \end{cases} \quad (20)$$

Note that p , $\underline{p}(\eta)$, λ , and σ will be determined as functions of m_0 later.

(II) Next, we obtain a candidate for a value function $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ consistent with the above strategy. From the above strategy, $V(\eta)$ for $\eta \in (0, B]$ can be written as a function of $V(0)$ as follows. First, for $\eta \in (0, p\bar{q}]$,

$$V(\eta) = \varepsilon\eta + \frac{m_0}{k} \left(a\frac{\eta}{p} + \beta V(0) \right) + \left(1 - \frac{m_0}{k} \right) \beta V(\eta)$$

holds. Thus

$$V(\eta) = A(m_0) \left(a\frac{\eta}{p} + \beta V(0) \right) + Z(m_0)\eta, \quad \text{for } \eta \in (0, p\bar{q}], \quad (21)$$

where $A(m_0) = \frac{m_0}{k-(k-m_0)\beta}$ and $Z(m_0) = \frac{k\varepsilon}{k-(k-m_0)\beta}$. Note that $A(m_0) < 1$. Similarly, $V(\eta)$ for $\eta \in (p\bar{q}, B]$ is written as:

$$V(\eta) = A(m_0)(a\bar{q} + \beta V(\eta - p\bar{q})) + Z(m_0)\eta, \quad \text{for } \eta \in (p\bar{q}, \infty). \quad (22)$$

Next, since an agent without money always chooses to be a seller, then $V(0)$ is determined by

$$V(0) = \frac{1-m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta V(\eta) \frac{f(\eta)}{1-m_0} d\eta \right] + \left(1 - \frac{1-m_0}{k} \right) \beta V(0). \quad (23)$$

(III) Below, we focus on equilibria with $V(0) = 0$ and obtain (p, λ, σ) as functions of m_0 . First, we decompose $\eta \geq 0$ into an multiple of $p\bar{q}$ and a residual; that is, $\eta = np\bar{q} + \iota$, where n is a nonnegative integer and ι is a nonnegative real number less than $p\bar{q}$. Then, by (21) and (22),

$$\begin{aligned} V(np\bar{q} + \iota) &= \frac{aA(m_0)}{1-\beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^n \left[\bar{q} - (1-\beta A(m_0)) \frac{\iota}{p} \right] \right\} \\ &+ Z(m_0) \left\{ \frac{n(1-\beta A(m_0)) - \beta A(m_0) + (\beta A(m_0))^{n+1}}{(1-\beta A(m_0))^2} p\bar{q} + \frac{1 - (\beta A(m_0))^{n+1}}{1-\beta A(m_0)} \iota \right\} \end{aligned} \quad (24)$$

holds. On the other hand, by (20) and (23),

$$(1-m_0)\bar{c} = \beta \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \int_{(0, p\bar{q}]} \eta f(\eta) d\eta = \beta \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \left(\frac{2}{3} \lambda p^3 \bar{q}^3 + \frac{1}{2} \sigma p^2 \bar{q}^2 \right) \quad (25)$$

holds.

Below, we obtain (p, λ, σ) as functions of m_0 . First, $1-m_0 = \int_{(0, B]} f d\eta$ can be written as follows:

$$1-m_0 = \int_{(0, p\bar{q}]} f d\eta = \lambda p^2 \bar{q}^2 + \sigma p \bar{q}. \quad (26)$$

Since the total amount of money the agents have is equal to M , the following equation must be satisfied:

$$M = \int_{(0, p\bar{q}]} \eta f d\eta = \frac{2}{3} \lambda p^3 \bar{q}^3 + \frac{1}{2} \sigma p^2 \bar{q}^2. \quad (27)$$

By (25), (26), (27), we obtain

$$p = \frac{Ma\beta A(m_0)}{(1-m_0)\bar{c} - M\beta Z(m_0)}, \quad (28)$$

$$\lambda = \frac{3(2M - p\bar{q}(1-m_0))}{p^3\bar{q}^3}, \quad (29)$$

$$\sigma = \frac{2(-3M + 2p\bar{q}(1-m_0))}{p^2\bar{q}^2}. \quad (30)$$

Suppose

$$\varepsilon < \frac{(1-\beta)\bar{c}}{M\beta}. \quad (31)$$

Then, $p \leq \frac{B}{\bar{q}}$ is satisfied if and only if

$$m_0 \leq \bar{m}_0 = -\frac{1}{2} \left(-1 + \frac{k(1-\beta)}{\beta} + \frac{Md}{B} \right) + \sqrt{\frac{1}{4} \left(-1 + \frac{k(1-\beta)}{\beta} + \frac{Md}{B} \right)^2 + \frac{k(1-\beta)}{\beta} - \frac{Mk}{\bar{c}} \varepsilon}. \quad (32)$$

It is verified $\bar{m}_0 \in (0, 1)$. We can also show that $m_0 \leq \bar{m}_0$ implies $p > 0$. Hereafter, we focus on m_0 satisfying (32).

(IV) Next, we check the optimality of the specified strategy.

(i) The optimality of the strategy of an agent with money holding $\eta > 0$:

First, we show that there exists a $\underline{p}(\eta) \geq p$ in (19). If $\eta \in (0, p\bar{q}]$, then by (24),

$$aq + \beta V(\eta - p_s q) = \left(a - a\beta A(m_0) \frac{p_s}{p} - \beta p_s Z(m_0) \right) q + \beta \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \eta$$

holds. Thus if

$$a - a\beta A(m_0) \frac{p_s}{p} - \beta p_s Z(m_0) \geq 0$$

holds, then she clearly chooses the maximum amount she can buy, and otherwise she chooses $q_b = 0$. Let

$$\underline{p}(\eta) = \frac{ap}{\beta(aA(m_0) + pZ(m_0))}, \quad \text{for } \eta \in (0, p\bar{q}]. \quad (33)$$

Then, (19) is optimal for $\eta \in (0, p\bar{q}]$. Suppose

$$\varepsilon \leq \frac{(1-\beta)\bar{c}(1-\bar{m}_0)}{M\beta}, \quad (34)$$

then (28) and (32) imply that $\underline{p}(\eta) \geq p$ holds. Hereafter, we focus on ε satisfying (34). (31) clearly holds. Similar arguments apply to the case of $\eta \in (p\bar{q}, B]$.

Next, we check an incentive for an agent with $\eta > 0$ to become a buyer instead of becoming a seller and offering (p', q') . By (19) and (33), for any $p' > \frac{ap}{\beta(aA(m_0)+pZ(m_0))}$, no buyer accepts such an offer on the equilibrium, and then the value is the same as that of an offer $(p'', 0)$, where $p'' \leq \frac{ap}{\beta(aA(m_0)+pZ(m_0))}$. Therefore, we can restrict our attention to (p', q') such that $p' \in \left[0, \frac{ap}{\beta(aA(m_0)+pZ(m_0))}\right]$ and $q' \in [0, \bar{q}]$ (provided $p'q' \leq B - \eta$). By (19), the value of becoming a seller and offering (p', q') is

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta \tilde{V}(\eta, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right] + \left(1 - \frac{1 - m_0}{k} \right) \beta V(\eta),$$

where

$$\tilde{V}(\eta, \eta') = \begin{cases} V(\eta + \eta'), & \text{if } \eta' \leq p'q', \\ V(\eta + p'q'), & \text{if } \eta' > p'q'. \end{cases} \quad (35)$$

On the other hand, when she becomes a buyer, the value is $V(\eta)$. Thus the difference is

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta \left(\tilde{V}(\eta, \eta') - V(\eta) \right) \frac{f(\eta')}{1 - m_0} d\eta' \right] - (1 - \beta)V(\eta). \quad (36)$$

Below, we show

$$V(\eta + \eta') - V(\eta) \leq \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \eta', \quad \text{for } \eta' \in (0, p\bar{q}]. \quad (37)$$

First, there exists a unique nonnegative integer n such that $np\bar{q} \leq \eta < (n+1)p\bar{q}$. There are two cases: (a) $\eta + \eta' < (n+1)p\bar{q}$ and (b) $\eta + \eta' \geq (n+1)p\bar{q}$. In case (a), by (24), (28), (32), (34), and $\beta A(m_0) < 1$,

$$\begin{aligned} V(\eta + \eta') - V(\eta) &= \left[(\beta A(m_0))^n \left(\frac{aA(m_0)}{p} - \frac{\beta A(m_0)Z(m_0)}{1 - \beta A(m_0)} \right) + \frac{Z(m_0)}{1 - \beta A(m_0)} \right] \eta' \\ &\leq \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \eta'. \end{aligned}$$

In case (b), by (24), (28), (32), (34) and $\beta A(m_0) < 1$,

$$\begin{aligned}
& V(\eta + \eta') - V(\eta) \\
&= (\beta A(m_0))^n \left\{ \left(aA(m_0) - \frac{p\beta A(m_0)Z(m_0)}{1 - \beta A(m_0)} \right) \right. \\
&\quad \left. \times \left[(n+1)(1 - \beta A(m_0))\bar{q} + \beta A(m_0)\frac{\eta'}{p} - (1 - \beta A(m_0))\frac{\eta}{p} \right] \right\} + \frac{Z(m_0)}{1 - \beta A(m_0)}\eta' \\
&\leq \left[(\beta A(m_0))^n \left(\frac{aA(m_0)}{p} - \frac{\beta A(m_0)Z(m_0)}{1 - \beta A(m_0)} \right) + \frac{Z(m_0)}{1 - \beta A(m_0)} \right] \eta' \\
&\leq \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \eta'.
\end{aligned}$$

The fourth line is obtained by $\eta \geq (n+1)p\bar{q} - \eta'$. This completes the proof of (37).

(24), (35), and (37) imply

$$\tilde{V}(\eta, \eta') - V(\eta) \leq \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \eta', \quad \text{for } \eta' \in (0, p\bar{q}].$$

Then, the first term of (36) is less than or equal to

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \left(\frac{aA(m_0)}{p} + Z(m_0) \right) \eta' \frac{f(\eta')}{1 - m_0} d\eta' \right].$$

This is equal to zero by the first equality of (25), and thus (36) is non-positive and she becomes a buyer.

(ii) The optimality of the strategy of an agent without money:

By the construction, an agent without money is indifferent between a buyer and a seller. Thus she has an incentive to be a seller. As in the latter part of (i), we restrict our attention to offers (p', q') such that $p' \in \left[0, \frac{ap}{\beta(aA(m_0) + pZ(m_0))} \right]$ and $q' \in [0, \bar{q}]$ (provided $p'q' \leq B$). By (19) and (24), the value of offering (p', q') is

$$\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0, p\bar{q}]} \beta \tilde{V}(0, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right], \quad (38)$$

where \tilde{V} is defined in (35). If $p'q' \geq p\bar{q}$, $\tilde{V}(0, \eta') = V(\eta')$ for any $\eta' \in (0, p\bar{q}]$. Then, (17) is

the same for all $p'q' \geq p\bar{q}$, and therefore the offer (p, \bar{q}) is optimal. If $p'q' \leq p\bar{q}$,

$$\begin{aligned} \int_{(0, p\bar{q}]} \tilde{V}(0, \eta') f(\eta') d\eta' &= \int_{(0, p'q']} V(\eta') f(\eta') d\eta' + \int_{(p'q', p\bar{q}]} V(p\bar{q}) f(\eta') d\eta' \\ &\leq \int_{(0, p'q']} V(\eta') f(\eta') d\eta' + \int_{(p'q', p\bar{q}]} V(\eta') f(\eta') d\eta' \\ &= \int_{(0, p\bar{q}]} \tilde{V}(0, p\bar{q}) f(\eta') d\eta', \end{aligned}$$

where the inequality is obtained by (24) and (35). Then, the offer (p, \bar{q}) is optimal. This completes the proof of (IV).

(V) Finally, we check $f(\eta) \geq 0$ for all $\eta \in (0, p\bar{q}]$. Since f is linear, it suffices to show $f(0) \geq 0$ and $f(p\bar{q}) \geq 0$. By (29) and (30),

$$f(0) = \sigma = \frac{2(-3M + 2p\bar{q}(1 - m_0))}{p^2\bar{q}^2}$$

and

$$f(p\bar{q}) = 2\lambda p\bar{q} + \sigma = \frac{2(3M - p\bar{q}(1 - m_0))}{p^2\bar{q}^2}$$

hold. A sufficient condition for $f(0) \geq 0$ and $f(p\bar{q}) \geq 0$ is clearly

$$\frac{3}{2} \leq \frac{p\bar{q}(1 - m_0)}{M} \leq 3.$$

By (28), this is equivalent to

$$\frac{3}{2} \leq \frac{m_0(1 - m_0)a\beta\bar{q}}{(1 - m_0)(k - (k - m_0)\beta)\bar{c} - M\beta k\varepsilon} \leq 3.$$

By (32), (34), and the assumption $d \leq 3$, the second inequality is always satisfied. Also, by (34), the first inequality is satisfied if

$$m_0 \geq \frac{3k(1 - \beta)}{\beta(2d - 3)} \quad (39)$$

holds. Therefore, there exists a continuum of m_0 satisfying (32) and (39) whenever $\bar{m}_0 > \frac{3k(1 - \beta)}{\beta(2d - 3)}$. By (32) and $d \geq \frac{3}{2}$, this is equivalent to

$$\frac{1}{B} < \frac{2(\beta(2d - 3) - 3k(1 - \beta))}{3M\beta(2d - 3)}.$$

Setting $\underline{\beta} = \frac{3k}{3(k-1)+2d}$, and for any $\beta \in (\underline{\beta}, 1)$, setting $\underline{B} = \frac{3M\beta(2d-3)}{2(\beta(2d-3)-3k(1-\beta))}$, we can show that for any $\beta \in (\underline{\beta}, 1)$ and for any $B > \underline{B}$, there exists a continuum of m_0 satisfying (32) and (39). Indeed, $\underline{\beta} < 1$ follows from the assumption $d > \frac{3}{2}$.

The stationary condition is expressed as follows:

$$m_0 \frac{1 - m_0}{k} \int_{(\eta, p\bar{q}]} \frac{f(\eta')}{1 - m_0} d\eta' = \int_{(\eta, p\bar{q}]} f(\eta') \frac{m_0}{k}, \quad \text{for } \eta \in [0, p\bar{q}],$$

where the LHS is the outflow from $[0, \eta]$, while the RHS is the inflow into $[0, \eta]$. Nevertheless, this is automatically satisfied. This concludes that for any $\beta \in (\underline{\beta}, 1)$ and for any $B > \underline{B}$, there exists a continuum of stationary equilibria if ε is so small that (34) holds, which means that any stationary equilibrium satisfies the commodity-money refinement in the sense of Zhou. ■

References

- [1] Kazuya Kamiya and Takashi Shimizu. Stationary monetary equilibria with strictly increasing value functions and non-discrete money holdings distributions: An indeterminacy result. mimeo, 2010.