Supplement to "Dynamic Auction Markets with Fiat Money"

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1 Walrasian Markets with Lotteries

This note is supplemental material to Kamiya and Shimizu [1]. We verify that the value function in Section 8 indeed satisfies the Bellman equation.

Recall that a lottery for buyers is defined by the terms of trade $\ell_b = (p_b, \lambda_b)$ and a lottery for sellers is defined by the terms of trade $\ell_s = (p_s, \lambda_s)$. Each agent solves the following problem:

$$\begin{aligned} \max_{q_b, q_s} \; & q_b \lambda_b u - q_s \lambda_s c + \gamma V(\eta') \\ \text{s.t. } & q_b, q_s \in \mathcal{R}_+, && (1) \\ & \min\{q_b, q_s\} = 0, && (2) \\ & q_b \lambda_b \leq 1, && (3) \\ & q_s \lambda_s \leq 1, && (4) \\ & \eta' = \eta - q_b p_b + q_s p_s, && (5) \\ & q_b p_b \leq \eta. && (6) \end{aligned}$$

Solving the above, individual demand and supply functions $q_b(\eta; \ell_b, \ell_s)$, $q_s(\eta; \ell_b, \ell_s)$ are obtained. Let $Q_b(\ell_b, \ell_s)$ and $Q_s(\ell_b, \ell_s)$ be the aggregate demand for buyers' and sellers' lotteries. An equilibrium in spot Walrasian market is defined by (ℓ_b, ℓ_s) satisfying

$$\begin{aligned} & \lambda_b Q_b(\ell_b, \ell_s) = \lambda_s Q_s(\ell_b, \ell_s), \\ & p_b Q_b(\ell_b, \ell_s) = p_s Q_s(\ell_b, \ell_s), \\ & \max\{\lambda_b, \lambda_s\} = 1. \end{aligned}$$

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In equilibrium,

$$\frac{\lambda_b}{\lambda_s} = \frac{p_b}{p_s} \tag{7}$$

must hold. In what follows, we restrict our attention to (ℓ_b, ℓ_s) satisfying (7).

In case of $1 = \lambda_b \geq \lambda_s$, the candidates for individual demand and supply functions are as follows:

$$q_b^*(\eta) = \begin{cases} 0 & \text{if } \eta < \bar{\eta}, \\ \min\left\{1, \frac{\eta}{p_b}\right\} & \text{if } \eta > \bar{\eta}, \end{cases} \tag{8}$$

$$q_s^*(\eta) = \begin{cases} \frac{p_b-\eta}{p_s} & \text{if } \eta < \bar{\eta}, \\ 0 & \text{if } \eta > \bar{\eta}, \end{cases} \tag{9}$$

where $\bar{\eta} \in (0, p_b)$. As for an agent with $\bar{\eta}$, he randomizes between buying $\bar{\eta}/p_b$ and selling $(p_b - \bar{\eta})/p_s$. From these functions, we obtain the following continuous value function

$$V(\eta) = \begin{cases} -\frac{p_b-\eta}{p_s}\lambda_s c + \gamma V(p_b) & \text{if } \eta < \bar{\eta}, \\ \frac{\eta}{p_b}u + \gamma V(0) & \text{if } \bar{\eta} \leq \eta < p_b, \\ u + \gamma V(\eta - p_b) & \text{if } \eta \geq p_b. \end{cases}$$

$\bar{\eta}$ must satisfy

$$-\frac{p_b - \bar{\eta}}{p_s}\lambda_s c + \gamma V(p_b) = \frac{\bar{\eta}}{p_b}u + \gamma V(0).$$

Then,

$$V(np_b + \iota) = \begin{cases} \frac{1}{1-\gamma}\left\{u - \gamma^n\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\iota}{p_b}c\right]\right\} & \text{if } 0 \leq \iota < \bar{\eta}, \\ \frac{1}{1-\gamma}\left\{u - \gamma^n\left[\frac{(1+\gamma-\gamma^2)u+\gamma c}{1+\gamma} - (1-\gamma)\frac{\iota}{p_b}u\right]\right\} & \text{if } \bar{\eta} \leq \iota < p_b, \end{cases} \tag{10}$$

and

$$\bar{\eta} = \frac{\gamma u - c}{(1+\gamma)(u-c)}p_b \tag{11}$$

hold. Clearly, $\bar{\eta} \in (0, p_b)$.

In case of $1 = \lambda_s > \lambda_b$, the candidate for individual demand and supply functions are as follows:

$$q_b^*(\eta) = \begin{cases} 0 & \text{if } \eta < \bar{\eta}, \\ \min\left\{\frac{p_s}{p_b}, \frac{\eta}{p_b}\right\} & \text{if } \eta > \bar{\eta}, \end{cases} \tag{12}$$

$$q_s^*(\eta) = \begin{cases} \frac{p_s-\eta}{p_s} & \text{if } \eta < \bar{\eta}, \\ 0 & \text{if } \eta > \bar{\eta}, \end{cases} \tag{13}$$

where $\bar{\eta} \in (0, p_s)$. As for an agent with $\bar{\eta}$, he randomizes between buying $\bar{\eta}/p_b$ and selling $(p_s - \bar{\eta})/p_s$. From these functions, we obtain the following continuous value function:

$$V(\eta) = \begin{cases} -\frac{p_s - \eta}{p_s} c + \gamma V(p_s) & \text{if } \eta < \bar{\eta}, \\ \frac{\eta}{p_b} \lambda_b u + \gamma V(0) & \text{if } \bar{\eta} \leq \eta < p_s, \\ \frac{p_s}{p_b} \lambda_b u + \gamma V(\eta - p_s) & \text{if } \eta \geq p_s. \end{cases}$$

$\bar{\eta}$ must satisfy

$$-\frac{p_s - \bar{\eta}}{p_s} c + \gamma V(p_s) = \frac{\bar{\eta}}{p_b} \lambda_b u + \gamma V(0).$$

Then,

$$V(np_b + \iota) = \begin{cases} \frac{1}{1-\gamma} \left\{ u - \gamma^n \left[\frac{u+c}{1+\gamma} - (1-\gamma) \frac{\iota}{p_s} c \right] \right\} & \text{if } 0 \leq \iota < \bar{\eta}, \\ \frac{1}{1-\gamma} \left\{ u - \gamma^n \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma) \frac{\iota}{p_s} u \right] \right\} & \text{if } \bar{\eta} \leq \iota < p_s, \end{cases} \tag{14}$$

and

$$\bar{\eta} = \frac{\gamma u - c}{(1+\gamma)(u-c)} p_s \tag{15}$$

hold. Clearly, $\bar{\eta} \in (0, p_b)$.

2 Optimality

2.1 Case of $1 = \lambda_b \geq \lambda_s$

In this section we show that V defined in (10) and (11), a candidate for the value function, indeed satisfies Bellman equation and (q_b^*, q_s^*) in (8) and (9) is an optimal policy. To be more precise, we define

$$\hat{V}(\mathbf{q}; \eta) = q_b \lambda_b \lambda_b u - q_s \lambda_s c + \gamma V(\eta') \tag{16}$$

where $\mathbf{q} = (q_b, q_s)$, η' is subject to (5), and V is defined in (10) and (11). We also define

$$V^*(\eta) = \max_{\mathbf{q} \text{ s.t. (1)-(6)}} \hat{V}(\mathbf{q}; \eta).$$

Below, we will show that

$$V(\eta) = V^*(\eta) = \hat{V}(\mathbf{q}^*; \eta) \tag{17}$$

holds, where $\mathbf{q}^* = (q_b^*, q_s^*)$ given in (8) and (9).

In this case, the constraints on q_b and q_s are written as

$$q_b \le \min\left\{1, \frac{\eta}{p_b}\right\},$$
$$q_s \le \frac{1}{\lambda_s}.$$

Moreover, by (7), the latter is clearly equivalent to

$$q_s p_s \le p_b.$$

2.1.1 $\eta \in [0, \bar{\eta})$

For $q_b \in [0, \eta/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b u + \gamma V(\eta - q_b p_b)\\ &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b}{p_b}c\right]\right\}\\ &= q_b(u - \gamma c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma(u+c)}{1-\gamma^2} + \gamma\frac{\eta}{p_b}c.\end{aligned}$$

Then, $\mathbf{q}_{111} = (\eta/p_b, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [0, (\bar{\eta} - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \gamma V(\eta + q_s p_s)\\ &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s}{p_b}c\right]\right\}\\ &= -(1-\gamma)q_s\lambda_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma(u+c)}{1-\gamma^2} + \gamma\frac{\eta}{p_b}c.\end{aligned}$$

Then, $\mathbf{q}_{112} = (0, 0)$ is the unique maximizer on the region.

For $a_b = 0$ and $q_s \in [(\bar{\eta} - \eta)/p_s, (p_b - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1-\gamma^2} - (1-\gamma)\frac{\eta + q_s p_s}{p_b}u\right]\right\}\\ &= q_s\lambda_s(\gamma u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma\frac{\eta}{p_b}u.\end{aligned}$$

Then, there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{113} = (0, (p_b - \eta)/p_s)$.

For $q_b = 0$ and $q_s \in [(p_b - \eta)/p_s, p_b/p_s]$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - p_b}{p_b}c\right]\right\}\\ &= -(1-\gamma^2)q_s\lambda_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^2(u+c)}{1-\gamma^2} + \gamma^2\frac{\eta - p_b}{p_b}c.\end{aligned}$$

Then, $\mathbf{q}_{114} = (0, (p_b - \eta)/p_s)$ is the unique maximizer on the region.

By (11), it is verified

$$\begin{aligned}&\hat{V}(\mathbf{q}_{114}) > \hat{V}(\mathbf{q}_{111}) > \hat{V}(\mathbf{q}_{112}),\\ &\lim_{\mathbf{q}\uparrow\mathbf{q}_{113}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{114}).\end{aligned}$$

Then, $\mathbf{q}_{114} = \mathbf{q}^*$ and (17) hold.

2.1.2 $\eta \in [\bar{\eta}, p_b)$

For $q_b \in ((\eta - \bar{\eta})/p_b, \eta/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b}{p_b}c\right]\right\}\\ &= q_b(u - \gamma c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma(u+c)}{1-\gamma^2} + \gamma\frac{\eta}{p_b}c.\end{aligned}$$

Then, $\mathbf{q}_{121} = (\eta/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in [0, (\eta - \bar{\eta})/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b}{p_b}u\right]\right\}\\ &= (1-\gamma)q_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma\frac{\eta}{p_b}u.\end{aligned}$$

Then, $\mathbf{q}_{122} = ((\eta - \bar{\eta})/p_b, 0)$ is the unique maximizer on the region.

For $p_b = 0$ and $q_s \in [0, (p_b - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s}{p_b}u\right]\right\}\\ &= q_s\lambda_s(\gamma u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma\frac{\eta}{p_b}u.\end{aligned}$$

Then, there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{123} = (0, (p_b - \eta)/p_s)$.

For $q_b = 0$ and $q_s \in [(p_b - \eta)/p_s, (p_b + \bar{\eta} - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - p_b}{p_b}c\right]\right\} \\ &= -(1-\gamma^2)q_s\lambda_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^2(u+c)}{1-\gamma^2} + \gamma^2\frac{\eta - p_b}{p_b}c.\end{aligned}$$

Then, $\mathbf{q}_{124} = (0, (p_b - \eta)/p_s)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [(p_b + \bar{\eta} - \eta)/p_s, p_b/p_s]$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - p_b}{p_b}u\right]\right\} \\ &= q_s\lambda_s(\gamma^2 u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^2\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^2\frac{\eta - p_b}{p_b}u.\end{aligned}$$

If $\theta > 1/\gamma^2$, then $\mathbf{q}_{125} = (0, p_b/p_s)$ is the unique maximizer on the region. If $\theta = 1/\gamma^2$, $\mathbf{q}_{125} = (0, q_s)$ for any $q_s \in [(p_b + \bar{\eta} - \eta)/p_s, p_b/p_s]$ is a maximizer on the region. If $\theta < 1/\gamma^2$, then $\mathbf{q}_{125} = (0, (p_b + \bar{\eta} - \eta)/p_s)$ is the unique maximizer on the region.

By (11), it is verified

$$\begin{aligned}&\hat{V}(\mathbf{q}_{121}) > \hat{V}(\mathbf{q}_{122}), \\ &\lim_{\mathbf{q}\uparrow\mathbf{q}_{123}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{124}), \\ &\hat{V}(\mathbf{q}_{121}) \geq \hat{V}(\mathbf{q}_{124}) \quad \text{"="} \text{ holds if and only if } \eta = \bar{\eta}, \\ &\hat{V}(\mathbf{q}_{121}) > \hat{V}(\mathbf{q}_{125}).\end{aligned}$$

Then $\mathbf{q}_{121} = \mathbf{q}^*$ and (17) hold. Moreover, any randomization between $\mathbf{q}_{121}$ and $\mathbf{q}_{124}$ is optimal if and only if $\eta = \bar{\eta}$.

2.1.3 $\eta \in [np_b, np_b + \bar{\eta})$ for $n \geq 1$

For $q_b \in ((\eta - (n-1)p_b - \bar{\eta})/p_b, 1]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n-1}\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - (n-1)p_b}{p_b}c\right]\right\} \\ &= q_b(u - \gamma^n c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^n(u+c)}{1-\gamma^2} + \gamma^n\frac{\eta - (n-1)p_b}{p_b}c.\end{aligned}$$

Then, $\mathbf{q}_{131} = (1, 0)$ is the unique maximizer on the region.

For $q_b \in ((\eta - np_b)/p_b, (\eta - (n-1)p_b - \bar{\eta})/p_b]$ and $p_s = 0$,

$$
\begin{aligned}
\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n-1}\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - (n-1)p_b}{p_B}u\right]\right\} \\
&= (1-\gamma^n)q_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma^n\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^n\frac{\eta-(n-1)p_b}{p_b}u.
\end{aligned}
$$

Then, $\mathbf{q}_{132} = ((\eta - (n-1)p_b - \bar{\eta})/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in [0, (\eta - np_b)/p_b]$ and $p_s = 0$,

$$
\begin{aligned}
\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - np_b}{p_b}c\right]\right\} \\
&= q_b(u - \gamma^{n+1}) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}(u+c)}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_b}{p_b}c.
\end{aligned}
$$

Then, $\mathbf{q}_{133} = ((\eta - np_b)/p_b, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [0, (np_b + \bar{\eta} - \eta)/p_s)$,

$$
\begin{aligned}
\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - np_b}{p_b}c\right]\right\} \\
&= -(1-\gamma^{n+1})q_s\lambda_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}(u+c)}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_b}{p_b}c.
\end{aligned}
$$

Then, $\mathbf{q}_{134} = (0, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [(np_b + \bar{\eta} - \eta)/p_s, ((n+1)p_b - \eta)/p_s)$,

$$
\begin{aligned}
\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{(1+\gamma-\gamma^2) + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - np_b}{p_b}u\right]\right\} \\
&= q_s\lambda_s(\gamma^{n+1}u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_b}{p_b}u.
\end{aligned}
$$

If $\theta > 1/\gamma^{n+1}$, then there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{135} = (0, ((n+1)p_b - \eta)/p_s)$. If $\theta = 1/\gamma^{n+1}$, then $\mathbf{q}_{135} = (0, q_s)$ for any $q_s \in [(np_b + \bar{\eta} - \eta)/p_s, ((n+1)p_b - \eta)/p_s)$ is a maximizer on the region. If $\theta < 1/\gamma^{n+1}$, then $\mathbf{q}_{135} = (0, (np_b + \bar{\eta} - \eta)/p_s)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [((n+1)p_b - \eta)/p_s, p_b/p_s]$,

$$
\begin{aligned}
\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n+1}\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - (n+1)p_b}{p_b}c\right]\right\} \\
&= -(1-\gamma^{n+2})q_s\lambda_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+2}(u+c)}{1-\gamma^2} + \gamma^{n+2}\frac{\eta - (n+1)p_b}{p_b}c.
\end{aligned}
$$

Then, $\mathbf{q}_{136} = (0, ((n+1)p_b - \eta)/p_s)$ is the unique maximizer on the region.

By (11), it is verified

$$
\begin{aligned}
&\hat{V}(\mathbf{q}_{131}) > \hat{V}(\mathbf{q}_{132}) > \hat{V}(\mathbf{q}_{133}) \geq \hat{V}(\mathbf{q}_{134}),\\
&\theta > \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \lim_{\mathbf{q}\uparrow\mathbf{q}_{135}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{136}),\\
&\theta \leq \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \hat{V}(\mathbf{q}_{131}) > \hat{V}(\mathbf{q}_{135}),\\
&\hat{V}(\mathbf{q}_{131}) > \hat{V}(\mathbf{q}_{136}).
\end{aligned}
$$

Then $\mathbf{q}_{131} = \mathbf{q}^*$ and (17) hold.

2.1.4 $\eta \in [np_b + \bar{\eta}, (n+1)p_b)$ for $n \geq 1$

For $q_b \in ((\eta - np_b)/p_b, 1]$ and $q_s = 0$,

$$
\begin{aligned}
\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n-1}\left[\frac{(1+\gamma-\gamma^2)+\gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - (n-1)p_b}{p_b}u\right]\right\}\\
&= (1-\gamma^n)q_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma^n\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^n \frac{\eta - (n-1)p_b}{p_b}u.
\end{aligned}
$$

Then, $\mathbf{q}_{141} = (1, 0)$ is the unique maximizer on the region.

For $q_b \in ((\eta - np_b - \bar{\eta})/p_b, (\eta - np_b)/p_b]$,

$$
\begin{aligned}
\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - np_b}{p_b}c\right]\right\}\\
&= q_b(u - \gamma^{n+1}c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}(u+c)}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_b}{p_b}c.
\end{aligned}
$$

Then, $\mathbf{q}_{142} = ((\eta - np_b)/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in [0, (\eta - np_b - \bar{\eta})/p_b]$,

$$
\begin{aligned}
\hat{V} &= q_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{(1+\gamma-\gamma^2)u+\gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - np_b}{p_b}u\right]\right\}\\
&= (1-\gamma^{n+1})q_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_b}{p_b}u.
\end{aligned}
$$

Then, $\mathbf{q}_{143} = ((\eta - np_b - \bar{\eta})/p_b, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [0, ((n+1)p_b - \eta)/p_s)$,

$$
\begin{aligned}
\hat{V} &= -q_s \lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{(1+\gamma-\gamma^2)u+\gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - np_b}{p_b}u\right]\right\}\\
&= q_s\lambda_s(\gamma^{n+1}u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_b}{p_b}u.
\end{aligned}
$$

If $\theta > 1/\gamma^{n+1}$, then there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{144} = (0, ((n+1)p_b - \eta)/p_s)$. If $\theta = 1/\gamma^{n+1}$, then $\mathbf{q}_{144} = (0, q_s)$ for any $q_s \in [0, ((n+1)p_b - \eta)/p_s)$ is a maximizer on the region. If $\theta < 1/\gamma^{n+1}$, then $\mathbf{q}_{144} = (0, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [((n+1)p_b - \eta)/p_s, ((n+1)p_b + \bar{\eta} - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n+1}\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - (n+1)p_b}{p_b}c\right]\right\} \\ &= -(1-\gamma^{n+2})q_s\lambda_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+2}(u+c)}{1-\gamma^2} + \gamma^{n+2}\frac{\eta - (n+1)p_b}{p_b}c.\end{aligned}$$

Then, $\mathbf{q}_{145} = (0, ((n+1)p_b - \eta)/p_s)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [((n+1)p_b + \bar{\eta} - \eta)/p_s, p_b/p_s]$,

$$\begin{aligned}\hat{V} &= -q_s\lambda_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n+1}\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - (n+1)p_b}{p_b}u\right]\right\} \\ &= q_s\lambda_s(\gamma^{n+2}u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+2}\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^{n+2}\frac{\eta - (n+1)p_b}{p_b}u.\end{aligned}$$

If $\theta > 1/\gamma^{n+2}$, then $\mathbf{q}_{146} = (0, p_b/p_s)$ is the unique maximizer on the region. If $\theta = 1/\gamma^{n+2}$, then $\mathbf{q}_{146} = (0, q_s)$ for any $q_s \in [((n+1)p_b + \bar{\eta} - \eta)/p_s, p_b/p_s]$ is a maximizer on the region. If $\theta < 1/\gamma^{n+2}$, then $\mathbf{q} = (0, ((n+1)p_b + \bar{\eta} - \eta)/p_s)$ is the unique maximizer.

By (11), it is verified

$$\begin{aligned}&\hat{V}(\mathbf{q}_{141}) > \hat{V}(\mathbf{q}_{142}) > \hat{V}(\mathbf{q}_{143}), \\ &\theta > \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \lim_{\mathbf{q}\uparrow\mathbf{q}_{144}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{145}), \\ &\theta \le \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \hat{V}(\mathbf{q}_{141}) > \hat{V}(\mathbf{q}_{144}), \\ &\hat{V}(\mathbf{q}_{141}) > \hat{V}(\mathbf{q}_{145}), \\ &\hat{V}(\mathbf{q}_{141}) > \hat{V}(\mathbf{q}_{146}).\end{aligned}$$

Then $\mathbf{q}_{141} = \mathbf{q}^*$ and (17) hold.

2.2 Case of $1 = \lambda_s > \lambda_b$

As in the previous section, we consider (16), where V is defined in (14) and (15), and we show that (17) holds, where $\mathbf{q}^* = (q_b^*, q_s^*)$ given in (12) and (13).

In this case constraints on q_b and q_s are written as

$$q_b \le \min\left\{\frac{1}{\lambda_b}, \frac{\eta}{p_b}\right\},$$
$$q_s \le 1.$$

Also, by (7), $q_b \le 1/\lambda_b$ is equivalent to

$$q_b p_b \le p_s.$$

2.2.1 $\eta \in [0, \bar{\eta})$

For $q_b \in [0, \eta/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b\lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b}{p_b}c\right]\right\} \\ &= q_b\lambda_b(u - \gamma c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma(u+c)}{1-\gamma^2} + \gamma\frac{\eta}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{211} = (\eta/p_b, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [0, (\bar{\eta} - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= = -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s}{p_s}c\right]\right\} \\ &= -(1-\gamma)q_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma(u+c)}{1-\gamma^2} + \gamma\frac{\eta}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{212} = (0, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [(\bar{\eta} - \eta)/p_s, (p_s - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s}{p_s}u\right]\right\} \\ &= (\gamma u - c)q_s + \frac{\gamma u}{1-\gamma} - \frac{\gamma\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma\frac{\eta}{p_s}u.\end{aligned}$$

Then, there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{213} = (0, (p_s - \eta)/p_s)$.

For $q_b = 0$ and $q_s \in [(p_s - \eta)/p_s, 1]$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - p_s}{p_s}c\right]\right\} \\ &= -(1-\gamma^2)q_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^2(u+c)}{1-\gamma^2} + \gamma^2\frac{\eta - p_s}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{214} = (0, (p_s - \eta)/p_s)$ is the unique maximizer on the region.

By (15), it is verified

$$\hat{V}(\mathbf{q}_{214}) > \hat{V}(\mathbf{q}_{211}) > \hat{V}(\mathbf{q}_{212}),$$
$$\lim_{\mathbf{q} \uparrow \mathbf{q}_{213}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{214}).$$

Then $\mathbf{q}_{214} = \mathbf{q}^*$ and (17) hold.

2.2.2 $\eta \in [\bar{\eta}, p_s)$

For $q_b \in ((\eta - \bar{\eta})/p_b, \eta/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b \lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b}{p_s} c\right]\right\} \\ &= q_b \lambda_b (u - \gamma c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma(u+c)}{1-\gamma^2} + \gamma \frac{\eta}{p_s} c.\end{aligned}$$

Then, $\mathbf{q}_{221} = (\eta/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in [0, (\eta - \bar{\eta})/p_b]$,

$$\begin{aligned}\hat{V} &= q_b \lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b}{p_s} u\right]\right\} \\ &= (1-\gamma) q_b \lambda_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma \frac{\eta}{p_s} u.\end{aligned}$$

Then, $\mathbf{q}_{222} = ((\eta - \bar{\eta})/p_b, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [0, (p_s - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s}{p_s} u\right]\right\} \\ &= q_s(\gamma u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma \frac{\eta}{p_s} u.\end{aligned}$$

Then, there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{223} = (0, (p_s - \eta)/p_s)$.

For $q_b = 0$ and $q_s \in [(p_s - \eta)/p_s, (p_s + \bar{\eta} - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - p_s}{p_s} c\right]\right\} \\ &= -(1-\gamma^2) q_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^2(u+c)}{1-\gamma^2} + \gamma^2 \frac{\eta - p_s}{p_s} c.\end{aligned}$$

Then, $\mathbf{q}_{224} = (0, (p_s - \eta)/p_s)$ is the unique maximizer on the region.

For$q_b = 0$ and $q_s \in [(p_s + \bar{\eta} - \eta)/p_s, 1]$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - p_s}{p_s}u\right]\right\} \\ &= q_s(\gamma^2 u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^2\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^2\frac{\eta - p_s}{p_s}u.\end{aligned}$$

If $\theta > 1/\gamma^2$, then $\mathbf{q}_{225} = (0, 1)$ is the unique maximizer on the region. If $\theta = 1/\gamma^2$, then $\mathbf{q}_{225} = (0, q_s)$ for any $q_s \in [(p_s + \bar{\eta} - \eta)/p_s, 1]$ is a maximizer on the region. If $\theta < 1/\gamma^2$, then $\mathbf{q}_{225} = (0, (p_s + \bar{\eta} - \eta)/p_s)$ is the unique maximizer on the region.

By (15), it is verified

$$\begin{aligned}&\hat{V}(\mathbf{q}_{221}) > \hat{V}(\mathbf{q}_{222}) \\ &\lim_{\mathbf{q}\uparrow\mathbf{q}_{223}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{224}), \\ &\hat{V}(\mathbf{q}_{221}) \geq \hat{V}(\mathbf{q}_{224}) \quad \text{"=" holds if and only if } \eta = \bar{\eta}, \\ &\hat{V}(\mathbf{q}_{221}) > \hat{V}(\mathbf{q}_{225})\end{aligned}$$

Then $\mathbf{q}_{221} = \mathbf{q}^*$ and (17) hold. Moreover, any randomization between $\mathbf{q}_{221}$ and $\mathbf{q}_{224}$ is optimal if and only if $\eta = \bar{\eta}$.

2.2.3 $\eta \in [np_s, np_s + \bar{\eta})$ for $n \geq 1$

For $q_b \in ((\eta - (n-1)p_s - \bar{\eta})/p_b, p_s/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b\lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n-1}\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - (n-1)p_s}{p_s}c\right]\right\} \\ &= q_b\lambda_b(u - \gamma^n c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^n(u+c)}{1-\gamma^2} + \gamma^n\frac{\eta - (n-1)p_s}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{231} = (p_s/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in ((\eta - np_s)/p_b, (\eta - (n-1)p_s - \bar{\eta})/p_s]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b\lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n-1}\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - (n-1)p_s}{p_s}u\right]\right\} \\ &= (1-\gamma^n)q_b\lambda_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma^n\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^n\frac{\eta - (n-1)p_s}{p_s}u.\end{aligned}$$

Then, $\mathbf{q}_{232} = ((\eta - (n-1)p_s - \bar{\eta})/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in [0, (\eta - np_s)/p_b]$ and $p_s = 0$,

$$\begin{aligned}\hat{V} &= q_b\lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - np_s}{p_s}c\right]\right\}\\ &= q_b\lambda_b(u - \gamma^{n+1}c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}(u+c)}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_s}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{233} = ((\eta - np_s)/p_b, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [0, (np_s + \bar{\eta} - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - np_s}{p_s}c\right]\right\}\\ &= -(1-\gamma^{n+1})q_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}(u+c)}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_s}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{234} = (0, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [(np_s + \bar{\eta} - \eta)/p_s, ((n+1)p_s - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^n\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - np_s}{p_s}u\right]\right\}\\ &= q_s(\gamma^{n+1}u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^{n+1}\frac{\eta - np_s}{p_s}u.\end{aligned}$$

If $\theta > 1/\gamma^{n+1}$, then there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{235} = (0, ((n+1)p_s - \eta)/p_s)$. If $\theta = 1/\gamma^{n+1}$, then $\mathbf{q}_{235} = (0, q_s)$ for any $q_s \in [(np_s + \bar{\eta} - \eta)/p_s, ((n+1)p_s - \eta)/p_s)$ is a maximizer on the region. If $\theta < 1/\gamma^{n+1}$, then $\mathbf{q}_{235} = (0, (np_s + \bar{\eta} - \eta)/p_s)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [((n+1)p_s - \eta)/p_s, 1]$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n+1}\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - (n+1)p_s}{p_s c}\right]\right\}\\ &= -(1-\gamma^{n+2})q_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+2}(u+c)}{1-\gamma^2} + \gamma^{n+2}\frac{\eta - (n+1)p_s}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{236} = (0, ((n+1)p_s - \eta)/p_s)$ is the unique maximizer on the region.

By (15), it is verified

$$\begin{aligned}&\hat{V}(\mathbf{q}_{231}) > \hat{V}(\mathbf{q}_{232}) > \hat{V}(\mathbf{q}_{233}) > \hat{V}(\mathbf{q}_{234}),\\ &\theta > \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \lim_{\mathbf{q}\uparrow\mathbf{q}_{235}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{236}),\\ &\theta \le \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \hat{V}(\mathbf{q}_{234}) > \hat{V}(\mathbf{q}_{235}),\\ &\hat{V}(\mathbf{q}_{231}) > \hat{V}(\mathbf{q}_{236}).\end{aligned}$$

Then $\mathbf{q}_{231} = \mathbf{q}^*$ and (17) hold.

2.2.4 $\eta \in [np_s + \bar{\eta}, (n+1)p_s)$ for $n \geq 1$

For $q_b \in ((\eta - np_s)/p_b, p_s/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b\lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n-1}\left[\frac{(1+\gamma-\gamma^2)u+\gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_s p_s - (n-1)p_s}{p_s}u\right]\right\} \\ &= (1-\gamma^n)q_b\lambda_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma^n\left[(1+\gamma-\gamma^2)u+\gamma c\right]}{1-\gamma^2} + \gamma^n\frac{\eta-(n-1)p_s}{p_s}u.\end{aligned}$$

Then, $\mathbf{q}_{241} = (p_s/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in ((\eta - np_s - \bar{\eta})/p_b, (\eta - np_s)/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b\lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n}\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - np_s}{p_s}c\right]\right\} \\ &= q_b\lambda_b(u - \gamma^{n+1}c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}(u+c)}{1-\gamma^2} + \gamma^{n+1}\frac{\eta-np_s}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{242} = ((\eta - np_s)/p_b, 0)$ is the unique maximizer on the region.

For $q_b \in [0, (\eta - np_s - \bar{\eta})/p_b]$ and $q_s = 0$,

$$\begin{aligned}\hat{V} &= q_b\lambda_b u + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n}\left[\frac{(1+\gamma-\gamma^2)u+\gamma c}{1+\gamma} - (1-\gamma)\frac{\eta - q_b p_b - np_s}{p_s}u\right]\right\} \\ &= (1-\gamma^{n+1})q_b\lambda_b u + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}\left[(1+\gamma-\gamma^2)u+\gamma c\right]}{1-\gamma^2} + \gamma^{n+1}\frac{\eta-np_s}{p_s}u.\end{aligned}$$

Then, $\mathbf{q}_{243} = ((\eta - np_s - \bar{\eta})/p_b, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [0, ((n+1)p_s - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n}\left[\frac{(1+\gamma-\gamma^2)u+\gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - np_s}{p_s}u\right]\right\} \\ &= q_s(\gamma^{n+1}u - c) + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+1}\left[(1+\gamma-\gamma^2)u+\gamma c\right]}{1-\gamma^2} + \gamma^{n+1}\frac{\eta-np_s}{p_s}u.\end{aligned}$$

If $\theta > 1/\gamma^{n+1}$, then there is no maximum and the supremum is obtained as $\mathbf{q} \to \mathbf{q}_{244} = (0, ((n+1)p_s - \eta)/p_s)$. If $\theta = 1/\gamma^{n+1}$, then $\mathbf{q}_{244} = (0, q_s)$ for any $q_s \in [0, ((n+1)p_s - \eta)/p_s)$ is a maximizer on the region. If $\theta < 1/\gamma^{n+1}$, then $\mathbf{q}_{244} = (0, 0)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [((n+1)p_s - \eta)/p_s, ((n+1)p_s + \bar{\eta} - \eta)/p_s)$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n+1}\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - (n+1)p_s}{p_s}c\right]\right\} \\ &= -(1-\gamma^{n+1})q_s c + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+2}(u+c)}{1-\gamma^2} + \gamma^{n+2}\frac{\eta-(n+1)p_s}{p_s}c.\end{aligned}$$

Then, $\mathbf{q}_{245} = (0, ((n+1)p_s - \eta)/p_s)$ is the unique maximizer on the region.

For $q_b = 0$ and $q_s \in [((n+1)p_s + \bar{\eta} - \eta)/p_s, 1]$,

$$\begin{aligned}\hat{V} &= -q_s c + \frac{\gamma}{1-\gamma}\left\{u - \gamma^{n+1}\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\eta + q_s p_s - (n+1)p_s}{p_s}u\right]\right\} \\ &= (\gamma^{n+2}u - c)q_s + \frac{\gamma u}{1-\gamma} - \frac{\gamma^{n+2}\left[(1+\gamma-\gamma^2)u + \gamma c\right]}{1-\gamma^2} + \gamma^{n+2}\frac{\eta-(n+1)p_s}{p_s}u.\end{aligned}$$

If $\theta > 1/\gamma^{n+2}$, then $\mathbf{q} = \mathbf{q}_{246} = (0,1)$ is the unique maximizer on the region. If $\theta = q/\gamma^{n+2}$, then $\mathbf{q}_{246} = (0, q_s)$ for any $q_s \in [((n+1)p_s + \bar{\eta} - \eta)/p_s, 1]$ is a maximizer on the region. If $\theta < 1/\gamma^{n+2}$, then $\mathbf{q}_{246} = (0, ((n+1)p_s + \bar{\eta} - \eta)/p_s)$ is the unique maximizer on the region.

By (15), it is verified

$$\begin{aligned}&\hat{V}(\mathbf{q}_{241}) > \hat{V}(\mathbf{q}_{242}) > \hat{V}(\mathbf{q}_{243}), \\ &\theta > \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \lim_{\mathbf{q}\uparrow\mathbf{q}_{244}} \hat{V}(\mathbf{q}) = \hat{V}(\mathbf{q}_{245}), \\ &\theta \leq \frac{1}{\gamma^{n+1}} \;\Rightarrow\; \hat{V}(\mathbf{q}_{243}) > \hat{V}(\mathbf{q}_{244}), \\ &\hat{V}(\mathbf{q}_{241}) > \hat{V}(\mathbf{q}_{245}), \\ &\hat{V}(\mathbf{q}_{241}) > \hat{V}(\mathbf{q}_{246}).\end{aligned}$$

Then $\mathbf{q}_{241} = \mathbf{q}^*$ and (17) hold.

References

[1] Kazuya Kamiya and Takashi Shimizu. Dynamic auction markets with fiat money. mimeo., 2009.