Dynamic Auction Markets with Fiat Money*

Kazuya Kamiya[†]and Takashi Shimizu[‡]

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Abstract

In this paper, we consider various types of models where fiat money is used to facilitate trades, and investigate whether models with fiat money have the same equilibrium allocations. In particular, we investigate whether or not such models have a continuum of stationary equilibria. We first present a dynamic centralized auction market model with money, which has not been investigated in the literature. We compare its outcome with that of Walrasian models with cash-in-advance constraints, and show that the outcomes are very different; the Walrasian markets model has determinate stationary equilibria and the auction markets model has a continuum of stationary equilibria. Moreover, we also build models on decentralized auction markets and on decentralized markets with bargaining, and obtain results similar to those of the dynamic centralized auction markets model.

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[†] Faculty of Economics, University of Tokyo, Bunkyo-ku, Tokyo 113-0033 JAPAN (E-mail: kkamiya@e.u-tokyo.ac.jp)

[‡] Faculty of Economics, Kansai University, 3-3-35 Yamate-cho, Suita, Osaka 564-8680 JAPAN (E-mail: tshimizu@kansai-u.ac.jp)

1 Introduction

There are various types of models where fiat money is used to facilitate trades, e.g., Walrasian models with cash-in-advance constraints and money search models. In this paper we investigate whether models with fiat money have the same equilibrium allocations. In particular, we investigate whether or not such models have a continuum of equilibria. We first present a dynamic centralized auction markets model with fiat money, which has not been investigated in the literature. We compare its outcome with that of a Walrasian model with cash-in-advance constraints, and show that the outcomes are very different; the Walrasian markets model has determinate stationary equilibria and the auction markets model has a continuum of stationary equilibria. In other words, the latter model has indeterminate stationary equilibria.[1] Moreover, we also build models on decentralized auction markets and on decentralized markets with bargaining, and obtain results similar to those of the dynamic centralized auction markets model.

In monetary economics, some papers investigate whether non-Walrasian monetary models lead to the Walrasian outcome. Among others, Green and Zhou [3] find that the law of one price, a feature of Walrasian equilibria, can hold in a money search model.[2] However, they also show that there exists a continuum of stationary one-price equilibria in the model, while there typically exists only a finite number of these in Walrasian models with cash-in-advance constraints. Subsequently, real indeterminacy of stationary equilibria has been found in both specific and general search models with divisible fiat money. (See, for example, Kamiya and Shimizu [7], Matsui and Shimizu [13], and Zhou [17].) The following questions then arise: (i) whether indeterminacy results hold in decentralized markets models other than search models; (ii) whether there exists a centralized markets model with indeterminate equilibria; (iii) what is the logic behind this indeterminacy. We answer these questions by presenting the inde-

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[1] In the literature, indeterminacy means the multiplicity of equilibria from a given initial point. In this paper, we define indeterminacy differently as the existence of a continuum of stationary equilibria.

[2] Kamiya and Sato [5] show that an equilibrium with dispersed prices (a two-price equilibrium) also exists in the model.

terminacy results on a centralized auction markets model and a decentralized auction markets model, and and by discussing the logic behind the indeterminacy.

It is worthwhile noting that Lagos and Wright [12] present a search model, where agents are also allowed for periodic access to Walrasian markets, and they show that the stationary equilibrium is unique and thus determinate.[3] Their determinacy result is clearly due to the existence of Walrasian markets. Therefore, one may think that more realistic centralized markets, such as auction markets, lead to determinacy results. However, in this paper, we show that this is not true; a dynamic auction model has a continuum of stationary equilibria.

A few papers have studied decentralized auctions with fiat money in environments different from ours; Julien et. al. [4] present a model with indivisible money, and Galenianos and Kircher [2] and Dutu et. al. [1] present a model with divisible money, where a periodic access to Walrasian markets is allowed as in Lagos and Wright [12]. In all these models, there is generically a unique stationary equilibrium due to the indivisibility of money, or Walrasian markets, while there is a continuum of stationary equilibria in our environment. Thus, a question arises as to whether there is indeterminacy of stationary equilibria in decentralized markets, such as decentralized auction markets and decentralized markets with bargaining, with periodic access to centralized auction markets. We discuss this problem in Section 8, and find that this is not so; such markets have a continuum of stationary equilibria.

We now turn to describe our model. We build a dynamic general equilibrium model with fiat money and a finite number of perishable goods. Fiat money is intrinsically worthless, and is used as a medium of exchange. The value of money is endogenously determined. For each good, there is a centralized market, where goods are traded by the double auction using fiat money. Each agent can produce a good that she cannot consume but is consumed by other agents. All agents have the same utility function and production cost, which are common knowledge to the agents. In each period, the

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[3] Rocheteau and Wright [15] compare three models with money: a search model, a price posting model, and a price taking model. As in Lagos and Wright's model, agents also have periodic access to Walrasian markets besides the markets above. They show that the models lead to different outcomes.

agent can visit only one centralized market. For example, at period t, if an agent wishes to obtain money, then she visits the market of her production good and in period $t+1$ she buys her consumption good using the money obtained at t.[4] We compare the equilibrium with the Walrasian equilibrium with cash-in-advance constraints, i.e., the case that equilibria are determined at the price at which demand is equal to supply and the expenditure of each agent is constrained by the amount of money she holds. We show that the set of equilibrium allocations with auction markets is a continuum, i.e., indeterminate, while that of the Walrasian markets model is a singleton. Therefore, the sets of equilibria do not coincide.

The equilibria is indeterminate due to an identity in the system of equilibrium conditions. The two different types of equilibria correspond to the following natures of monetary trades:

1. the amount of money the sellers obtain is always equal to that the buyers pay even out of equilibria, and
2. the amount of money the sellers obtain is not necessarily equal to that the buyers pay.

The auction market and random matching market with money are classified as of the former type and the Walrasian market is classified as of the latter type. In the former case, the market clearing condition for money is not an equation but an identity, because the demand for money is always equal to the supply even out of equilibria. Due to this identity, the number of variables is one more than the number of equations, and thus the set of equilibria is a continuum.

This paper is organized as follows. In Section 2, we present the common environment in this paper. In Section 3, we investigate a dynamic centralized auction markets model and show that there is a continuum of stationary equilibria. In Section 4, we consider a Walrasian markets model with cash-in-advance constraints, and show that the equilibrium is essentially unique. In Section 5, we extend the uniqueness result in Warlasin markets to the environment in which lotteries are traded. In Sections 6 and

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[4] Even without this participation constraint, we can obtain almost the same results in these cases. See Appendix B.

7, we investigate a model with decentralized auctions and a random matching model with bargaining, and obtain similar results for these cases. In Section 8, we investigate the logic behind the indeterminacy by classifying monetary trades into two types. In Section 9, we conclude the paper. Finally, in the Appendix, we show that similar results hold even when mixed strategy is introduced or when each agent can choose to be a seller and a buyer simultaneously.

2 The Environment and Equilibrium Concepts

In this section, we present the environment common throughout this paper. Time is discrete and denoted by $t = 1, 2, \ldots$. There is a continuum of agents whose measure is one. There are $\ell \geq 3$ types of agents with equal fractions and the same number of types of goods. We assume that goods are indivisible as in Green and Zhou [3]. Only one unit of indivisible and perishable good i can be produced by a type $i-1 \pmod{\ell}$ agent with production cost $c > 0$. A type i agent obtains utility $u > 0$ only when she consumes one unit of good i. Let $\theta = u/c$. In this production-consumption structure, there is *no double coincidence of wants*, which is common in the literature of money search models. (See, for example, Kiyotaki and Wright [11].) Let $\gamma \in (0, 1)$ be the discount factor. Moreover, we make the following assumption.

Assumption 1 $\theta > 1$ and $\gamma > 1/\theta$.

Since $\gamma > 1/\theta$ is equivalent to $\gamma u > c$, it implies that an agent gains when she produces her production good in this period and consumes her consumption good in the next period.

There is completely divisible and durable fiat money and its nominal stock is given by $M > 0$. The above exogenously given parameters are common knowledge among the agents, and therefore there is no informational asymmetry. For simplicity, we assume that each agent is under a *participation constraint*; she can join at most one market in each period. In other words, she must choose to be a seller or a buyer, or do nothing at the beginning of each period. However, we show that even without this participation constraint, we obtain almost the same results. This is discussed later in the following sections. (See Appendix B.)

We formulate models with various trading institutions as dynamic market games and compare their outcomes with those of Walrasian markets. We adopt the notion of a recursive competitive equilibrium as a Walrasian markets equilibrium, where the allocation is symmetric across the types, and the money holdings distribution is stationary. The counterpart of the recursive competitive equilibrium in dynamic market games is a stationary Markov perfect equilibrium, where the strategies are symmetric across the types and the money holdings distribution is stationary. It is typically hard to obtain all equilibria in such models. However, in order to show differences, it is sufficient to find an equilibrium different from the Walrasian equilibrium. Thus we focus on a special class of stationary Markov perfect equilibria.

Moreover, to fine-tune our result, we introduce the concept of robustness. We refer to an equilibrium as *robust to the money-holding-cost refinement* if it is still an equilibrium even when holding money incurs an infinitesimally small cost.[5] We show that the robust Walrasian equilibrium is unique. Note that, even in the case without the robustness, similar results can be obtained. (See Remarks 2 and 3 in the following sections.)

3 Centralized Auction Markets

In this section, we consider centralized auction markets. In each period, a centralized market is open for each good. At the jth market, good j is traded by the multi-person k-double auctions, where $k \in [0, 1]$ is exogenously given. The details of the auctions are given below. At the beginning of each period, each agent decides whether and which market to join. If a type i agent joins the $(i + 1)$th market, then she becomes a seller of her product, good i, while if she joins the ith market, then she becomes a buyer.

Moreover, we restrict our attention to the class of stationary equilibria that satisfy the following:

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[5] Wallace and Zhu [16] present another kind of robustness, the commodity-money refinement. We instead adapt the money-holding-cost refinement because of its tractability. Instead of introducing a value of consuming money, introducing a small cost of money holdings has only a temporal effect on agents' behavior and no effect on the strategy of stationary equilibria to which we restrict our attention.

- any agent chooses a pure strategy depending only on the current money holding,[6]
- all trades occur with the same price p on the equilibrium path, and
- the support of money holdings distributions is $\{0, p\}$.

In the k-double auction, a seller posts an ask price and a buyer posts a bid price. Since we confine our attention to the class specified above, it is sufficient to define the auction rules for the following three cases:

(i) all the sellers post the same ask price s^*, and all the buyers post the same bid price b^*,

(ii) all the agents but one seller follow (i), and this seller posts an ask price $\hat{s}$ that is different from s^*,

(iii) all the agents but one buyer follow (i), and this buyer posts an ask price $\hat{b}$ that is different from b^*.

Let $S, B \in [0, 1]$ be the measures of sellers and buyers, respectively. First, we consider case (i). Assume $s^* > b^*$, then no transaction occurs. Assume $s^* \leq b^*$. If $s^* \leq b^*$ and $S < B$, then the price is b^*, each seller can sell a good with probability one, and each buyer can buy a good with probability S/B. If $s^* \leq b^*$ and $S > B$, the price is s^*, each seller can sell a good with probability B/S , and each buyer can buy a good with probability one. If $s^* \leq b^*$ and $S = B$, the price is determined by $p = kb^* + (1 - k)s^*$, and all the agents can trade goods with probability one.

Next, in cases (ii) and (iii), the price is determined as in case (i), and the probabilities of trade for all agents except the single deviator are also determined as in case (i). In case (ii), the deviated seller can sell a good with probability one if $\hat{s} < s^*$, or $S < B$ and $\hat{s} \leq b^*$, and cannot sell a good otherwise. In case (iii), the deviated buyer can buy a good with probability one if $\hat{b} > b^*$, or $S > B$ and $\hat{b} \geq s^*$, and cannot buy a good otherwise.

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[6] Even when we allow for mixed strategies, we obtain a similar result. See Appendix A.

A (pure) Markov strategy is a function from the set of money holdings distributions R_+ to $(\{\sigma\} \times R_+) \cup (\{\beta\} \times R_+) \cup \{\nu\}$, where $(\sigma, s) \in \{\sigma\} \times R_+$ implies that an agent chooses to be a seller and posts an ask price s, $(\beta, b) \in \{\beta\} \times R_+$ implies that an agent chooses to be a buyer and bids a price b, and ν implies that an agent does nothing. Each agent maximizes the discounted sum of expected utility given the other agents' strategy. *A stationary Markov perfect equilibrium* is a pair of a Markov strategy and a money holdings distribution such that no agent has an incentive to deviate from the strategy and the distribution is stationary under the strategy.

Moreover, we restrict our attention to the equilibria in which all trades occur with a common price $p > 0$ and the money holdings distribution has a support only on $\{0, p\}$. Let $h = (h_0, h_1)$ be a money holdings distribution where h_n is the measure of agents with np.

As in the previous section, we require an equilibrium to be robust to the refinement by infinitesimally small money-holding-cost; if the values of η and $\bar{\eta}$ are the same and $\eta > \bar{\eta}$, then there is no incentive to hold η. Thus, if the value of holding p is larger than that of no money holding, then any distribution with the support $\{0, p\}$ is robust.

The following is a candidate for an equilibrium strategy:

- an agent with $\eta \in [0, p)$ chooses to be a seller and posts an ask price p , and
- an agent with $\eta \in [p, \infty)$ chooses to be a buyer and posts a bid price η.

We consider the case of $h_0 \leq \frac{1}{2}$. This implies that the measure of buyers is larger than or equal to that of sellers, and therefore an agent with $\eta \in (p, \infty)$ would not necessarily win even if she had chosen to be a buyer.

The stationarity of money holdings distributions is expressed as follows. Since there is excess demand in each market, the measure of agents who can buy is $(1 - h_0)\frac{h_0}{1-h_0}$ and their money holdings become 0. On the other hand, since agents without money can always sell, the measure of agents who can sell is h_0 and their money holdings become p. Thus the stationarity of money holdings distribution at 0 is

$$(1 - h_0)\frac{h_0}{1 - h_0} = h_0,$$

where the LHS is the inflow at 0 and the RHS is the outflow at 0. Similarly, the stationarity of money holdings distribution at p is

$$h_0 = (1 - h_0)\frac{h_0}{1 - h_0}.$$

Both of them are identities and thus any (h_0, h_1) satisfying $h_0 + h_1 = 1, h_0 \geq 0$, and $h_1 \geq 0$, is a stationary distribution.

Under the strategy, the Bellman equation is expressed as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta < p, \\ r\left(u + \gamma V(0)\right) + (1 - r)\gamma V(p), & \text{if } \eta = p, \\ u + \gamma V(\eta - p), & \text{if } \eta > p, \end{cases}$$

where

$$r = \frac{h_0}{1 - h_0}.$$

An agent with $\eta < p$ chooses to be a seller and can always sell a good. Thus, she obtains $-c + \gamma V(\eta + p)$. An agent with $\eta = p$ chooses to be a buyer and can buy a good with probability r. Thus, she obtains $-r\left(u + \gamma V(0)\right) + (1 - r)\gamma V(p)$. An agent with $\eta > p$ chooses to be a buyer and can always buy a good. Thus, she obtains $u + \gamma V(\eta - p)$.

We decompose η into an integer multiple of p and a residual; that is, $\eta = np + \iota$, where n is a nonnegative integer and ι is a nonnegative real number less than p. Then, the value function is expressed as follows:

$$V(np + \iota) = \begin{cases} \frac{r\gamma u - (1-\gamma+r\gamma)c}{(1-\gamma)(1+r\gamma)}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{1+r\gamma}\left[(1 - r + r\gamma)u + r\gamma c\right]\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n}}{1+\gamma}(u + c)\right\}, & \text{if } \iota \neq 0. \end{cases} \tag{1}$$

Note that ι does not appear in the value function since such a fractional amount of money does not generate any value to the holder on the equilibrium.

We need to check the following incentives:

(i) incentive for an agent with $\eta \in [0, p)$ to be a seller instead of doing nothing,

(ii) incentive for an agent with $\eta \in [0, p)$ to be a seller instead of being a buyer,

(iii) incentive for a seller with $\eta \in [0, p)$ to post p instead of any other price,

(iv) incentive for an agent with $\eta \in [p, \infty)$ to be a buyer instead of doing nothing,

(v) incentive for an agent with $\eta \in [p, \infty)$ to be a buyer instead of being a seller, and

(vi) incentive for a buyer with $\eta \in [p, \infty)$ to bid η instead of bidding another price.

It is easily verified that (ii) is reduced to (i), and (iii), (iv), and (vi) are automatically satisfied. (i) is reduced to (I) below. As for (v), it is obvious that we only need to check (II) below, the incentive at $\eta = p$.

(I) $V(0) \geq 0$,

(II) $V(p) \geq -c + \gamma V(2p)$.

By (1), the following inequality is equivalent to (I):

$$r \geq \frac{1-\gamma}{\gamma(\theta-1)}. \tag{2}$$

Again, by (1), the following inequality is equivalent to (II):

$$r \geq \frac{\gamma\theta - 1}{(1+\gamma-\gamma^2)\theta - \gamma^2}. \tag{3}$$

Since we assume $\gamma > \frac{1}{\theta}$, the RHSs of (2) and (3) lie in $(0, 1)$. Thus, for any r such that

$$r \in \left[\max\left\{\frac{1-\gamma}{\gamma(\theta-1)}, \frac{\gamma\theta-1}{(1+\gamma-\gamma^2)\theta-\gamma^2}\right\}, 1\right], \tag{4}$$

there is a corresponding equilibrium, and moreover, the above interval is non-empty. Since $r = \frac{h_0}{1-h_0}$, (4) is equivalent to

$$h_0 \in \left[\max\left\{\frac{1-\gamma}{\gamma(\theta-2)+1}, \frac{\gamma\theta-1}{(1+2\gamma-\gamma^2)\theta-1-\gamma^2}\right\}, \frac{1}{2}\right].$$

From the above, we obtain the following theorem.

Theorem 1 Under Assumption 1, for any h_0 satisfying

$$h_0 \in \left[\max\left\{\frac{1-\gamma}{\gamma(\theta-2)+1}, \frac{\gamma\theta-1}{(1+2\gamma-\gamma^2)\theta-1-\gamma^2}\right\}, \frac{1}{2}\right],$$

there exists a stationary Markov perfect equilibrium with $p = \frac{M}{1-h_0}$. Moreover, the equilibria obtained in the above are robust to the refinement by infinitesimally small money-holding-cost.

4 Walrasian Markets

In this section we analyze a simple dynamic Walrasian market, and show that the equilibrium is unique. In each period a competitive market is open for each good. We define a version of a recursive competitive equilibrium, called a stationary Walrasian equilibrium. More precisely, in the equilibrium (a) each agent maximizes the discounted sum of utility stream for given prices of goods under the budget constraint and the cash-in-advance constraint, (b) the markets of goods clear, i.e., for each good, the measure of sellers is equal to that of buyers, (c) the money demand is equal to supply, and (d) the money holdings distribution and the prices of goods are stationary, i.e., time-invariant. Moreover, we require that (e) the equilibria is robust to the refinement by infinitesimally small money-holding-cost, i.e., if the values of money holdings η and $\bar{\eta}$ are the same and $\eta > \bar{\eta}$, then no one wants to hold η. That is, since holding money incurs infinitesimally small cost, agents discard $\eta - \bar{\eta}$ in this case.

We focus on stationary equilibria in which all agents with identical characteristics act similar and in which all of the ℓ types are symmetric. Thus, we seek for equilibria, where the prices of goods are the same. Let the common price be $p \in R_+$. For a given p, the behavior of an agent with money holding $\eta \in R_+$ is expressed in terms of a Bellman equation. If she can afford to buy a good, i.e., $\eta \geq p$, then she can choose either 'buy', 'sell', or 'do nothing'. Otherwise, the cash-in-advance constraint is binding and she can choose either 'sell' or 'do nothing'. Thus the Bellman equation is as follows:

$$V(\eta) = \begin{cases} \max\{u + \gamma V(\eta - p), -c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta \geq p, \\ \max\{-c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta < p, \end{cases} \tag{5}$$

where $u + \gamma V(\eta - p), -c + \gamma V(\eta + p)$, and $\gamma V(\eta)$ correspond to 'buy', 'sell', or 'do nothing', respectively. For a given p, the unique value function $V : R_+ \to R$ and the optimal policy correspondence $\phi : R_+ \to \{\beta, \sigma, \nu\}$ are obtained, where β, σ , and ν represent 'buy', 'sell', and 'do nothing', respectively. Let $\psi(\eta, x) \in [0, 1]$ be the proportion of agents with η choosing $x \in \{\beta, \sigma, \nu\}$, where $\sum_{x \in \{\beta,\sigma,\nu\}} \psi(\eta, x) = 1$. ψ is said to be consistent with ϕ, if $\psi(\eta, x) > 0$ implies $x \in \phi(\eta)$ for all $\eta \in R_+$. For example, let $\phi(\eta) = \{\beta, \sigma\}$ and $\psi(\eta, \beta) = \psi(\eta, \sigma) = 1/2$, then clearly ψ is consistent with ϕ, and half of the agents with money holding η buy a good and move from η to $\eta - p$ and the other half sell a good and move from η to $\eta + p$.

Let $\mathcal{B}$ be the Borel σ-algebra on $[0, \infty)$, and F be a money holdings distribution. That is, F is a probability measure on $\mathcal{B}$.

Then, a robust stationary Walrasian equilibrium is defined as follows.

Definition 1 $\langle p, F, V, \phi, \psi \rangle$ is said to be a *robust stationary Walrasian equilibrium* if

(i) given p, V satisfies (5),

(ii) $\phi : R_+ \to \{\beta, \sigma, \nu\}$ is the optimal policy correspondence associated with V,

(iii) ψ is consistent with ϕ,

(iv) F is stationary, [7]

(v) $\int_{n=0}^{\infty} \eta dF = M$,

(vi) $\psi(\eta, x)$ is a measurable function of η, and

$$\int \psi(\eta, \beta) dF = \int \psi(\eta, \sigma) dF$$

holds,

(vii) $\langle p, F, V, \phi, \psi \rangle$ is robust to the refinement by infinitesimally small money-holding-cost. That is, for all $\bar{\eta} \in R_+$, $F(\{\eta | V(\eta) = V(\bar{\eta}) \text{ and } \eta > \bar{\eta}\}) = 0$ holds.

(vi) is the market clearing condition for goods; namely, the measure of buyers is equal to that of sellers. Note that, by Walras' law, money demand is equal to money supply if (vi) is satisfied. (vii) means that if the values of η and $\bar{\eta}$ are the same and $\eta > \bar{\eta}$, no one wants to have η, because holding money incurs an infinitesimally small cost. Note that even without (vii) we can obtain almost the same results. (See Remarks 2 and 3 below.)

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[7] More precisely, F is stationary under the transition function defined below:

$$\begin{aligned} T(\eta, \{\eta + p\}) &= \psi(\eta, \sigma), \\ T(\eta, \{\eta - p\}) &= \psi(\eta, \beta), \\ T(\eta, \{\eta\}) &= \psi(\eta, \nu). \end{aligned}$$

Note that for the other $A \in \mathcal{B}$, $T(\eta, A)$ is naturally defined from the above.

Obviously, $p = 0$ is not consistent with the incentive of sellers. Suppose $p > 0$ is an equilibrium price. An agent with $\eta \geq p$ obviously chooses to be a buyer. Thus, $\eta \geq 2p$ is a transient state. Suppose $\gamma V(\eta+p) - c > \gamma V(\eta)$. Then, an agent with $\eta < p$ chooses to be a seller. Then, the Bellman equation becomes as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta \in [0, p), \\ u + \gamma V(\eta - p), & \text{otherwise.} \end{cases}$$

The value function is obtained as follows:

$$V(\eta) = \begin{cases} \frac{\gamma u - c}{1-\gamma^2}, & \text{if } \eta \in [0, p), \\ \frac{u - \gamma c}{1-\gamma^2}, & \text{if } \eta \in [p, 2p), \\ \vdots & \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^n}{1+\gamma}(u+c)\right\}, & \text{if } \eta \in [np, (n+1)p), \\ \vdots & \end{cases} \tag{6}$$

Note that $\gamma V(\eta + p) - c > \gamma V(\eta)$ holds for $\eta \in [0, p)$ if $\gamma > \frac{1}{\theta}$ is satisfied. Since V is a step function with a step p, (vii) implies that all agents have an amount that is an integer multiple of p. Thus, it is straightforward to show that the stationary Walrasian equilibrium allocation is achieved when half of the agents have no money and their value is $V(0) = \frac{\gamma u - c}{1-\gamma^2}$, and the other half have p and their value is $V(p) = \frac{u - \gamma c}{1-\gamma^2}$. Indeed, the measure of sellers is equal to that of buyers and thus the market clearing condition holds. The money holding distribution is clearly stationary. From the above, we obtain the following theorem.

Theorem 2 Under Assumption 1, there exists a unique robust stationary Walrasian equilibrium $\langle p, F, V, \phi, \psi \rangle$ characterized by

(I) $p = 2M$,

(II) $F(\{0\}) = 1/2$ and $F(\{p\}) = 1/2$,

(III) V is given by (6),

(IV)

$$\phi(\eta) = \begin{cases} \{\sigma\}, & \text{if } \eta \in [0, p), \\ \{\beta\}, & \text{if } \eta \in [p, \infty), \end{cases}$$

(V)

$$\psi(\eta, x) = 1, \quad \text{iff} \begin{cases} \eta < p & \text{and} \quad x = \sigma, \quad \text{or} \\ \eta \geq p & \text{and} \quad x = \beta. \end{cases}$$

Remark 1 Since we are assuming the indivisibility of goods, the market clearing condition is that the measure of buyers, $1 - h_0$, is equal to that of sellers, h_0. Thus one may think that the uniquness result immediately follows from the indivisibility. In Section 5, we introduce the notion of Walrasian equilibrium with lotteries, where markets clear even if $-h_0 = h_0$ does not hold. Nevertheless, we can show that $1 - h_0 = h_0$ always holds in stationary equilibria, and thus the stationary equilibrium is unique.

Remark 2 Even if we do not impose the robustness condition (viii), (V, ϕ, ψ) in equilibria are the same as in the above theorem, while any (p, F) satisfying $\int \eta dF = M$, $F([0, p)) = 1/2$, $F([p, 2p)) = 1/2$, and

$$F([0, \eta]) = F([p, p + \eta]), \quad \forall \eta \in [0, p),$$

can be an equilibrium price and money holdings distribution. In the equilibria, the value of $\eta \in [0, p)$ is always $\frac{\gamma u - c}{1 - \gamma^2}$ and the value of $\eta \in [p, 2p)$ is always $\frac{u - \gamma c}{1 - \gamma^2}$. That is, the distribution of value is the same as in the above theorem, and thus the equilibrium is essentially unique even without the robustness. (See Kamiya and Shimizu [6].)

We now compare the auction markets outcome derived in the previous section with the robust stationary Walrasian equilibrium outcome derived in this section. When h_0 is fixed at 1/2 in the value function (1), we obtain $V(0) = \frac{\gamma u - c}{1 - \gamma^2}$ and $V(p) = \frac{u - \gamma c}{1 - \gamma^2}$. In this case, half of the agents have the former value and the rest have the latter value, i.e., exactly the same as in the case of the Walrasian markets. As shown in Theorem 1, there are other stationary equilibria for $h_0 \neq \frac{1}{2}$ in the auction markets.

Corollary 1 The set of outcomes in the centralized auction markets does not coincide with that in the Walrasian markets.

Remark 3 Even without the robustness, the same results can be obtained, since the Walrasian equilibrium is essentially unique even without the robustness. (See Remark

2.) Note that the same argument applies to the decentralized auction markets discussed in Section 6 and the random matching and bargaining discussed in Section 7.

5 Walrasian Markets with Lotteries

As discussed in the previous section, in the simple dynamic Walrasian market, markets do not clear if $1-h_0 \neq h_0$. In this section, we introduce the notion of *Walrasian markets with lotteries*, and show that the equililibrium is unique as in the simple Walrasian market model in the previous section.

There are two kinds of lotteries: a lottery for buyers and a lottery for sellers. The lottery for buyers is characterized by $\ell_b = (p_b, \lambda_b)$, where p_b is the price of lottery and λ_b is the probability of obtaining the good when the buyer has one unit of the lottery. If a buyer has q_b units of lottery, then the probability of obtaining the good is $q_b\lambda_b$. Of course, $q_b\lambda_b \leq 1$ must be satisfied. Similarly, the lottery for sellers is characterized by $\ell_s = (p_s, \lambda_s)$, where p_s is the price of lottery and λ_s is the probability of selling the good when the seller has one unit of the lottery. If a seller has q_s units of lottery, then the probability of selling the good is $q_s\lambda_s$. Of course, $q_s\lambda_s \leq 1$ must be satisfied. We confine our attention to the following lotteries. If the measure of buyers is larger than that of sellers, then the sellers can sell the good with probability one, i.e., $\lambda_s = 1$. (λ_b will be determined later.) Similarly, if the measure of sellers is larger than that of buyers, then the buyers can buy the good with probability one, i.e., $\lambda_b = 1$. (λ_s will be determined later.)

As in the previous sections, we assume that an agent chooses either to be a seller or a buyer, i.e., $\min\{q_b, q_s\} = 0$. Thus, an agent solves the following problem:

$$
\begin{aligned}
\max_{q_b,q_s}\ & q_b\lambda_b u - q_s\lambda_s c + \gamma V(\eta') \\
\text{s.t. } & q_b, q_s \in \mathcal{R}_+, \\
& \min\{q_b, q_s\} = 0, \\
& q_b\lambda_b \leq 1, \\
& q_s\lambda_s \leq 1, \\
& \eta' = \eta - q_b p_b + q_s p_s, \\
& q_b p_b \leq \eta,
\end{aligned}
$$

where $q_b p_b \leq \eta$ is the cash-in-advance constraint. Then, the demand and supply functions $q_b(\eta; \ell_b, \ell_s)$ and $q_s(\eta; \ell_b, \ell_s)$ can be obtained. Let F on $[0, \infty)$ be a stationary money holdings distribution. Then $Q_b(\ell_b, \ell_s) = \int q_b(\eta; \ell_b, \ell_s)dF$ and $Q_s(\ell_b, \ell_s) = \int q_s(\eta; \ell_b, \ell_s)dF$ are the aggregate demand and the aggregate supply of lotteries, respectively. (ℓ_b, ℓ_s) is said to be an equilibrium if the following conditions are satisfied:

$$
\begin{aligned}
& \lambda_b Q_b(\ell_b, \ell_s) = \lambda_s Q_s(\ell_b, \ell_s), \\
& p_b Q_b(\ell_b, \ell_s) = p_s Q_s(\ell_b, \ell_s), \\
& \max\{\lambda_b, \lambda_s\} = 1.
\end{aligned}
$$

If $Q_b(\ell_b, \ell_s) > Q_s(\ell_b, \ell_s)$, then, by $\max\{\lambda_b, \lambda_s\} = 1$, $\lambda_s = 1$ holds. Thus $Q_s(\ell_b, \ell_s)$ is the supply of goods and is equal to the measure of sellers. $Q_s(\ell_b, \ell_s)$ is divided into lotteries such that the measure of winners (buyers who can consume goods) is equal to $Q_s(\ell_b, \ell_s)$, i.e., $\lambda_b Q_b(\ell_b, \ell_s) = Q_s(\ell_b, \ell_s)$. Similarly, if $Q_b(\ell_b, \ell_s) < Q_s(\ell_b, \ell_s)$, then, by $\max\{\lambda_b, \lambda_s\} = 1$, $\lambda_b = 1$ holds. Thus, $Q_b(\ell_b, \ell_s)$ is the demand of goods and is equal to the measure of buyers. $Q_b(\ell_b, \ell_s)$ is divided into lotteries such that the measure of winners (sellers who can sell goods) is equal to $Q_b(\ell_b, \ell_s)$, i.e., $Q_b(\ell_b, \ell_s) = \lambda_s Q_s(\ell_b, \ell_s)$. The second condition is the market clearing condition: the aggregate value of demand is equal to the aggregate value of supply. By the first and second conditions,

$$
\frac{\lambda_b}{\lambda_s} = \frac{p_b}{p_s} \tag{7}
$$

holds in an equilibrium. Thus, in what follows, we restrict our attention to (ℓ_b, ℓ_s) satisfying (7).

First, we consider the case of $1 = \lambda_b \geq \lambda_s$. In this case, we will show that the following functions are individual demand and supply functions:

$$q_b^*(\eta) = \begin{cases} 0 & \text{if } \eta < \bar{\eta}, \\ \min\left\{1, \frac{\eta}{p_b}\right\} & \text{if } \eta > \bar{\eta}, \end{cases}$$

$$q_s^*(\eta) = \begin{cases} \frac{p_b - \eta}{p_s} & \text{if } \eta < \bar{\eta}, \\ 0 & \text{if } \eta > \bar{\eta}, \end{cases}$$

where $\bar{\eta} \in (0, p_b)$. As for an agent with $\bar{\eta}$, he randomizes between buying $\bar{\eta}/p_b$ and selling $(p_b - \bar{\eta})/p_s$. From these functions, the following continuous value function is obtained:

$$V(\eta) = \begin{cases} -\frac{p_b - \eta}{p_s}\lambda_s c + \gamma V(p_b) & \text{if } \eta < \bar{\eta}, \\ \frac{\eta}{p_b} u + \gamma V(0) & \text{if } \bar{\eta} \leq \eta < p_b, \\ u + \gamma V(\eta - p_b) & \text{if } \eta \geq p_b. \end{cases}$$

$\bar{\eta}$ must satisfy

$$-\frac{p_b - \bar{\eta}}{p_s}\lambda_s c + \gamma V(p_b) = \frac{\bar{\eta}}{p_b} u + \gamma V(0).$$

Then,

$$V(np_b + \iota) = \begin{cases} \frac{1}{1-\gamma}\left\{u - \gamma^n\left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\iota}{p_b}c\right]\right\} & \text{if } 0 \leq \iota < \bar{\eta}, \\ \frac{1}{1-\gamma}\left\{u - \gamma^n\left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\iota}{p_b}u\right]\right\} & \text{if } \bar{\eta} \leq \iota < p_b, \end{cases}$$

and

$$\bar{\eta} = \frac{\gamma u - c}{(1+\gamma)(u-c)} p_b.$$

Clearly, $\bar{\eta} \in (0, p_b)$. It is also verified that V satisfies the Bellman equation and (q_b^*, q_s^*) is an optimal policy.[8]

Given q_b^* and q_s^*, any state except $\{0\}$ and $\{p_b\}$ is transient. Then, the stationary distribution is $F(\{0\}) = F(\{p_b\}) = 1/2$.

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[8] Since it is tedious, the verification is omitted and relegated to Kamiya and Shimizu [10].

Next, we consider the case of $1 = \lambda_s > \lambda_b$. In this case, we will show that the following functions are individual demand and supply functions:

$$q_b^*(\eta) = \begin{cases} 0 & \text{if } \eta < \bar{\eta}, \\ \min\left\{\frac{p_s}{p_b}, \frac{\eta}{p_b}\right\} & \text{if } \eta > \bar{\eta}, \end{cases}$$

$$q_s^*(\eta) = \begin{cases} \frac{p_s - \eta}{p_s} & \text{if } \eta < \bar{\eta}, \\ 0 & \text{if } \eta > \bar{\eta}, \end{cases}$$

where $\bar{\eta} \in (0, p_s)$. As for an agent with $\bar{\eta}$, he randomizes between buying $\bar{\eta}/p_b$ and selling $(p_s - \bar{\eta})/p_s$. From these functions, we obtain the following continuous value function

$$V(\eta) = \begin{cases} -\frac{p_s - \eta}{p_s} c + \gamma V(p_s) & \text{if } \eta < \bar{\eta}, \\ \frac{\eta}{p_b} \lambda_b u + \gamma V(0) & \text{if } \bar{\eta} \leq \eta < p_s, \\ \frac{p_s}{p_b} \lambda_b u + \gamma V(\eta - p_s) & \text{if } \eta \geq p_s. \end{cases}$$

$\bar{\eta}$ must satisfy

$$-\frac{p_s - \bar{\eta}}{p_s} c + \gamma V(p_s) = \frac{\bar{\eta}}{p_b} \lambda_b u + \gamma V(0).$$

Then,

$$V(np_b + \iota) = \begin{cases} \frac{1}{1-\gamma}\left\{u - \gamma^n \left[\frac{u+c}{1+\gamma} - (1-\gamma)\frac{\iota}{p_s} c\right]\right\} & \text{if } 0 \leq \iota < \bar{\eta}, \\ \frac{1}{1-\gamma}\left\{u - \gamma^n \left[\frac{(1+\gamma-\gamma^2)u + \gamma c}{1+\gamma} - (1-\gamma)\frac{\iota}{p_s} u\right]\right\} & \text{if } \bar{\eta} \leq \iota < p_s, \end{cases}$$

and

$$\bar{\eta} = \frac{\gamma u - c}{(1+\gamma)(u-c)} p_s.$$

Clearly, $\bar{\eta} \in (0, p_b)$. It is also verified that V satisfies the Bellman equation and (q_b^*, q_s^*) is an optimal strategy.[9]

Given q_b^* and q_s^*, any state except $\{0\}$ and $\{p_s\}$ is transient. Then the stationary distribution is determined as $F(\{0\}) = F(\{p_s\}) = 1/2$. Thus by the equilibrium condition, $p_s = p_b$ holds.

In summary, the money holding distribution in a stationary equilibrium satisfies $F(\{0\}) = F(\{p\}) = 1/2$ for some $p > 0$. Moreover, the values at $\{0\}$ and $\{p\}$ are

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[9] The verification is relegated to Kamiya and Shimizu [10].

the same as those in Walrasian equilibria without lotteries. In other words, the real allocation in Walrasian equilibria is unique and the same as that in Walrasian equilibria without lotteries.

Theorem 3 A real allocation of any stationary equilibrium in Walrasian markets with lotteries is the same as that in Walrasian markets without lotteries. Moreover, any stationary equilibrium is robust.

6 Decentralized Auction Markets

In this section, we consider an economy, where trades take place in decentralized second price auction markets. Note that the model can be considered as an essentially dynamic version of Peters [14]-type models. We show that there is also a continuum of robust stationary equilibria, and therefore the set of robust equilibria does not coincide with that of Walrasian markets.

In each period, each agent chooses either to be a seller, be a buyer, or do nothing. Each seller opens a second-price auction market and posts a minimum bid of her second-price auction. After observing the distribution of posted minimum bids, each buyer simultaneously chooses the auction he participates in. After observing the number of other participants in the auction he participates in, he bids a price.

We focus on stationary Markov perfect equilibria in which all trades occur with the same price $p > 0$. To be more precise, *a stationary Markov perfect equilibrium* is a pair of a Markov strategy and a money holdings distribution such that the strategy is optimal and the distribution is stationary under the strategy. We also require the equilibrium to be robust.

We consider the following strategy:

- an agent with $\eta \in [0, p)$ chooses to be a seller and posts a minimum bid p,
- an agent with p chooses to be a buyer and always bids p,
- an agent with $\eta \in [p, \infty)$ chooses to be a buyer, and

- bids p if there is no other participant in the auction,
- bids η if there are other participants in the auction.

Moreover, we consider stationary equilibria where

(i) a stationary money holdings distribution is characterized by $h = (h_0, h_1)$, where h_n is the measure of agents with money holding np,

(ii) $h_0 \leq \frac{1}{2}$, and

(iii) every buyer randomizes with equal probabilities between auctions with the same minimum bid.

The equilibrium with (iii) is often investigated in directed search models, e.g., Peters [14]. In such an equilibrium, although the measure of the sellers is bigger than or equal to that of the buyers, a seller may get left out of the trade. To be more precise, let $r = \frac{1-h_0}{h_0} \geq 1$, and consider a finite economy with f sellers and rf buyers, where f is a positive integer. Then on the equilibrium path, the probability that a seller finds at least one buyer visiting her auction is $1 - ((f-1)/f)^{rf}$. This converges to $\alpha = 1 - e^{-r}$ as $f \to \infty$. We consider α as the probability that a seller can succeed in selling in our large economy. Similarly, the probability that a buyer succeeds in buying on the equilibrium path is $\frac{\alpha}{r}$.

The Bellman equation is as follows:

$$V(\eta) = \begin{cases} \alpha\left(-c + \gamma V(\eta + p)\right) + (1-\alpha)\gamma V(\eta), & \text{if } \eta < p, \\ \frac{\alpha}{r}\left(u + \gamma V(0)\right) + \left(1 - \frac{\alpha}{r}\right)\gamma V(p), & \text{if } \eta = p, \\ u + \gamma V(\eta - p), & \text{if } \eta > p. \end{cases}$$

An agent with $\eta < p$ chooses to be a seller and can sell a good with probability α. Thus, she obtains $\alpha\left(-c + \gamma V(\eta + p)\right) + (1-\alpha)\gamma V(\eta)$. An agent with $\eta = p$ chooses to be a buyer and can buy a good with probability $\frac{\alpha}{r}$. Thus, she obtains $\frac{\alpha}{r}\left(u + \gamma V(0)\right) + \left(1 - \frac{\alpha}{r}\right)\gamma V(p)$. An agent with $\eta > p$ chooses to be a buyer and can always buy a good. Thus, she obtains $u + \gamma V(\eta - p)$.

We decompose η into an integer multiple of p and a residual; that is, $\eta = np + \iota$, where n is a nonnegative integer and ι is a nonnegative real number less than p. Then, we obtain

$$V(np+\iota) = \\ \begin{cases} \frac{\gamma\alpha^2(u-c)-(1-\gamma)\alpha rc}{(1-\gamma)[(1-\gamma+\gamma\alpha)r+\gamma\alpha]}, & \text{if } \iota = 0,\ n=0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{(1-\gamma+\gamma\alpha)r+\gamma\alpha}\left\{[(r-\alpha)(1-\gamma+\gamma\alpha)+\gamma\alpha]\,u+\gamma\alpha^2 c\right\}\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n}}{1+\gamma\alpha}(u+\alpha c)\right\}, & \text{if } \iota \neq 0. \end{cases}$$

The incentive conditions are the same as (i)-(vi) proposed earlier in the centralized auction model. As in the case of the centralized auction model, we only need to check the following inequalities:

(I) $V(0) \geq 0$,

(II) $V(p) \geq -c + \gamma V(2p)$.

(I) is equivalent to

$$\gamma \geq \underline{\gamma}(r) = \frac{r}{(\theta-1)\alpha + r}.$$

Note that $\underline{\gamma}(r) \in (1/\theta, 1)$. Similarly, (II) is equivalent to

$$\begin{aligned} H(\gamma;r) = {} & \left[r(1-\alpha)\theta - \alpha(2-\alpha)\theta - \alpha^2\right]\gamma^2 \\ & + \left[-r(1-\alpha+\theta) + \alpha^2\theta + \alpha(1-\alpha)\right]\gamma + r + \alpha\theta \geq 0. \end{aligned}$$

Since

$$\begin{aligned} & H(1/\theta;r) > 0, \\ & H(1;r) < 0, \\ & \frac{\partial H(\gamma;r)}{\partial \gamma} < 0, \quad \forall \gamma \in [0,1], \end{aligned}$$

there exists $\overline{\gamma}(r) \in (1/\theta, 1)$ such that, for $\gamma \in (0,1)$,

$$\gamma \leq \overline{\gamma}(r) \Leftrightarrow H(\gamma;r) \geq 0.$$

By tedious calculations, we obtain

$$H\left(\underline{\gamma}(1);1\right) = \frac{\alpha^*(\theta^2-1)}{\left[\alpha^*\theta + (1-\alpha^*)\right]^2}\left\{(\alpha^*)^2(\theta-1) + (3\alpha^*-1)\right\},$$

where $\alpha^* = 1 - e^{-1}$. Since $\alpha^* > \frac{1}{3}$, it is verified that $H\left(\underline{\gamma}(1); 1\right) > 0$, and therefore $\overline{\gamma}(1) > \underline{\gamma}(1)$. The continuity of $\overline{\gamma}(r)$ and $\underline{\gamma}(r)$ at $r = 1$ implies there exist $\underline{\gamma}$, $\overline{\gamma}$, and $\overline{r}$ such that

$$\begin{aligned} &\frac{1}{\theta} < \underline{\gamma} < \overline{\gamma} < 1, \\ &\overline{r} > 1, \end{aligned}$$

and for any $\gamma \in (\underline{\gamma}, \overline{\gamma})$, there exists a stationary equilibrium with the specified strategy with any $r \in [1, \overline{r})$. Let $\overline{h}_0 = \frac{1}{\overline{r}+1}$, then we obtain the following result:

Theorem 4 There exist $\underline{\gamma}$, $\overline{\gamma}$, and $\overline{h}_0$ satisfying $\frac{1}{\theta} < \underline{\gamma} < \overline{\gamma} < 1$ and $\overline{h}_0 \in (0, \frac{1}{2})$ such that, for any given $\gamma \in (\underline{\gamma}, \overline{\gamma})$, a robust stationary Markov perfect equilibrium exists for any $h_0 \in \left(\overline{h}_0, \frac{1}{2}\right]$.

Since there is a continuum of equilibria with different allocations, we obtain the following theorem from the above.

Corollary 2 The set of outcomes in the decentralized auction markets does not coincide with that in the Walrasian markets.

7 Random Matching and Bargaining

In this section, we consider a decentralized markets model, where a term of trade is determined by random matching and bargaining, and show that there exits a continuum of robust stationary equilibria, and therefore, the set of robust equilibria does not coincide with that of Walrasian markets.

At the beginning of each period, each agent chooses to be a seller, to be a buyer, or to do nothing. Then with probability $\mu \in (0, 1]$, she is randomly matched with another active agent (i.e., an agent who has decided to be a seller or a buyer), and the type of partner is randomly determined. Recall that a type i agent consumes good i and produces good $i + 1$. If she is a type i seller (buyer), she can only trade when she meets a type $(i + 1)$ buyer (a type $(i - 1)$ seller). The bargaining proceeds as seller's take-it-or-leave-it offer protocol. We assume that a seller offers a price without knowing the partner's money holdings.

In this framework, we characterize a stationary Markov perfect equilibrium in which all trades occur with the same price $p > 0$. To be more precise, *a stationary Markov perfect equilibrium* is a pair of a Markov strategy and a money holdings distribution such that the strategy is optimal and the distribution is stationary under the strategy. Moreover, we restrict our attention to equilibria with the support of money holdings distribution $\{0, p\}$. Let $h = (h_0, h_1)$ be a money holdings distribution, where h_n is the measure of agents with money holding np. We also require the equilibrium to be robust.

A candidate for an equilibrium strategy is as follows:

- an agent with $\eta \in [0, p)$ chooses to be a seller and offers a price p, and
- an agent with $\eta \in [p, \infty)$ chooses to be a buyer and accepts p.

Under the strategy described above, the Bellman equation is as follows:

$$V(\eta) = \begin{cases} \mu\ell^{-1}h_1\left(-c + \gamma V(\eta + p)\right) + (1 - \mu\ell^{-1}h_1)\gamma V(\eta), & \text{if } \eta < p, \\ \mu\ell^{-1}h_0\left(u + \gamma V(\eta - p)\right) + (1 - \mu\ell^{-1}h_0)\gamma V(\eta), & \text{if } \eta \geq p. \end{cases}$$

A type i agent with $\eta < p$ chooses to be a seller. With probability $\mu\ell^{-1}h_1$, she meets a type $i+1$ agent with $\eta \geq p$ and obtains $-c + \gamma V(\eta + p)$, and with probability $(1 - \mu\ell^{-1}h_1)$, she does not meet an appropriate agent and obtains $\gamma V(\eta)$. A type i agent with $\eta \geq p$ chooses to be a buyer. With probability $\mu\ell^{-1}h_0$, she meets a type $i-1$ agent with $\eta < p$ and obtains $u + \gamma V(\eta - p)$, and with probability $(1 - \mu\ell^{-1}h_0)$, she does not meet an appropriate agent and obtains $\gamma V(\eta)$.

We decompose η into an integer multiple of p and a residual; that is, $\eta = np + \iota$, where n is a nonnegative integer and ι is a nonnegative real number less than p. Then, we obtain

$$V(np + \iota) = \frac{\mu\ell^{-1}}{1-\gamma}\left\{h_0 u - A^n \frac{1 - \gamma + \gamma\mu\ell^{-1}h_0}{1 - \gamma + \gamma\mu\ell^{-1}}(h_0 u + h_1 c)\right\},$$

where

$$A = \frac{\gamma \mu \ell^{-1} h_0}{1 - \gamma + \gamma \mu \ell^{-1} h_0}.$$

The incentive conditions are the same as in the centralized auctions model, i.e.,

(i) $V(0) \geq 0$,

(ii) $V(np) \geq \mu \ell^{-1} h_1 \left(-c + \gamma V((n+1)p)\right) + (1 - \mu \ell^{-1} h_1)\gamma V(np)$.

The former condition is equivalent to

$$\gamma \geq \frac{1}{1 + \mu \ell^{-1} h_0 (\theta - 1)}.$$

The latter condition is automatically satisfied since $A < 1$. Thus, we obtain the following theorem.

Theorem 5 Let $\underline{\gamma}_R = \frac{1}{1+\mu\ell^{-1}(\theta-1)}$. Then, for any given $\gamma \in \left(\underline{\gamma}_R, 1\right)$ and any $h_0 \in \left[\frac{1-\gamma}{\gamma\mu\ell^{-1}(\theta-1)}, 1\right)$, there exists a robust stationary Markov equilibrium with $p = \frac{M}{1-h_0}$.

Since there is a continuum of equilibria with different allocations while the equilibrium is unique in the Walrasian markets, we obtain the following theorem.

Corollary 3 The set of outcomes in the dynamic model of random matching and bargaining does not coincide with that in the Walrasian markets.

8 Real Indeterminacy of Stationary Equilibria in Monetary Models

In this section, we explore the logic behind Corollary 1. As shown in the previous sections, the auction markets have a continuum of stationary equilibrium allocations, while the Walrasian markets have the unique equilibrium allocation, and thus the outcomes do not coincide. Below, we show that there are two types of fundamental natures of monetary trades; one that has a continuum of stationary equilibrium and the other that has locally unique equilibria. A typical example of the former case is the auction market, while that of the latter case is the Walrasian market.

First, we confine our attention to the environment in Section 2. In the Walrasian markets, the price of goods is determined in the centralized markets. Thus, as shown in Section 4, it suffices to investigate money holdings distributions with a support expressed by $\{0, p\}$, where $p > 0$ is an equilibrium price. In auction markets, there may exist equilibrium money holdings distributions whose supports are not $\{0, p\}$. However, in order to show that the outcomes in the latter case includes the outcomes in the former case, we only need to focus on distributions with a support expressed by $\{0, p\}$. Let $h = (h_0, h_1)$ be a probability distribution on the support, where h_n is a measure of agent with money holding np.

Below, we compare the equilibrium conditions in the previous two sections. Limiting our analysis to the case that the measure of buyers is larger than or equal to that of sellers, the equilibrium condition for for the centrilized auction markets is as follows:

$$\begin{aligned} p &= \frac{M}{h_1}, \\ h_0 + h_1 &= 1, \\ h_1 \frac{h_0}{h_1} &= h_0, \\ h_0 &= h_1 \frac{h_0}{h_1}, \\ V(\eta) &= \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta < p, \\ r\left(u + \gamma V(0)\right) + (1 - r)\gamma V(p), & \text{if } \eta = p, \\ u + \gamma V(\eta - p), & \text{if } \eta > p. \end{cases} \end{aligned}$$

On the other hand, the equilibrium condition for Walrasian markets is as follows:

$$\begin{aligned} p &= \frac{M}{h_1}, \\ h_0 + h_1 &= 1, \\ h_1 \psi(p, \beta) &= h_0 \psi(0, \sigma), \\ h_0 \psi(0, \sigma) &= h_1 \psi(p, \beta), \\ V(\eta) &= \begin{cases} \gamma V(\eta + p) - c & \text{if } \eta \in [0, p), \\ \gamma V(\eta - p) + u & \text{otherwise}, \end{cases} \end{aligned}$$

where the third and the forth equations are the conditions for stationarity of money holdings. In the both systems, for a given (h_0, h_1), p and V are uniquely determined by

$p = \frac{M}{h_1}$ and the Bellman equations. As for (h_0, h_1), in the equilibrium condition for the auction markets, any (h_0, h_1) satisfying $h_0 + h_1 = 1$, $h_0 \geq 0$, and $h_1 \geq 0$, is a stationary distribution, since $h_1 \frac{h_0}{h_1} = h_0$ and $h_0 = h_1 \frac{h_0}{h_1}$ are identities. On the other hand, in the equilibrium condition for the Walrasian markets, (h_0, h_1) satisfying $h_0 + h_1 = 1$ is not necessarily a stationary distribution, since $h_1 \psi(p, \beta) = h_0 \psi(0, \sigma)$ is not an identity. (See the discussion below.) Thus, there is one degree of freedom in the former system, while the solution is determinate in the latter system.

To be more general, the two different types of equilibria correspond to the following natures of monetary trades:

(i) the amount of money the sellers obtain is always equal to that the buyers pay even out of equilibria, and

(ii) the amount of money the sellers obtain is not necessarily equal to that the buyers pay.

It is clear that the auction market is classified as the former type and the Walrasian market is classified as the latter type. Indeed, in the auction markets, the amount of money the sellers obtain is ph_0 and that the buyers pay is $ph_1 \frac{h_0}{h_1} = ph_0$. By this identity, any money holdings distribution is stationary, i.e., all (h_0, h_1) satisfying $h_0 + h_1 = 1$, $h_0 \geq 0$, and $h_1 \geq 0$ is stationarity, and the set of equilibria is a continuum. In the Walrasian markets, if p is very large, then all agents without money choose to be sellers, i.e., $\psi(0, \sigma) = 1$, and the amount of money the sellers obtain is ph_0. On the other hand, the amount of money the buyers pay, which is at most $p(1 - h_0)$, is not necessarily equal to ph_0.

In order to understand the logic of indeterminacy, we investigate more general environments, where a money holdings distribution is expressed as $h = (h_0, h_1, \ldots, h_N)$ for some positive integer N. Suppose the values of the other variables, besides h, are given. Note that these variables are determined by the Bellman equations for a given h; namely, the number of such variables are equal to the number of equations in the Bellman equations. Let I_n, a function of h, be a measure of agents whose money holdings are not np before trades and become np after trades, and O_n, a function of h, be a measure of agents whose money holdings are np before trades and become $n'p$ for some $n' \neq n$ after trades. In other words, I_n is the measure of agents in the inflow at np, and O_n is the measure of agents in the outflow at np. The stationary condition is expressed by $I_n = O_n$ for $n = 0, 1, \ldots, N$. Clearly, $\sum_{n=0}^{N} I_n - \sum_{n=0}^{N} O_n = 0$ always holds, i.e., this is an identity, since each agent, who belongs to an outflow at some n, should belong to an inflow at some n' and thus the total measure of agents in all inflows, expressed by $\sum_{n=0}^{N} I_n$, is equal to that in all outflows, expressed by $\sum_{n=0}^{N} O_n$.

Now suppose (i), i.e., the amount of money the sellers obtain is always equal to that of the buyers pay even out of equilibria. Then $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$ always holds, i.e., this is also an identity. Indeed, the total amount of money before trades, expressed by $\sum_{n=0}^{N} pnO_n$, is equal to the total amount of money after trades, expressed by $\sum_{n=0}^{N} pnI_n$, and thus $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$ always holds. Thus there are $N+1$ equations, $I_n = O_n$, $n = 0, 1, \ldots, N$, and two identities, $\sum_{n=0}^{N} I_n - \sum_{n=0}^{N} O_n = 0$ and $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$; i.e., the number of linearly independent equations among them is $N - 1$. On the other hand, $\sum_{n=0}^{N} h_n - 1 = 0$ is the other restriction and the number of such variables, $h_0, h_1, \ldots, h_N$, is $N + 1$. Therefore, there is at least one degree of freedom in the determination of stationary distribution. This leads to the real indeterminacy of stationary equilibria.

In decentralized markets models with divisible money, monetary trades are typically classified as of the first type. (See, for example, Green and Zhou [3], Kamiya and Shimizu [7], Matsui and Shimizu [13], and Zhou [17].) For example, suppose (a) a buyer randomly meets a seller, (b) then the buyer offers a price and the amount of good she wishes to buy, and (c) the seller decides whether to accept or to reject the offer. In this case, in each matching, the amount of money the buyer pays is always the same as that of the seller obtains even out of equilibria. Thus, in the economy, the total amount of money the buyers pay is always the same as that of the sellers obtain even out of equilibria. By the same logic as in the centralized economy, the equilibria are typically indeterminate. The only difference between decentralized markets models and centralized markets models of the first type is that equilibrium price dispersion might occur in decentralized markets models. (See Kamiya and Sato [5] and Matsui and Shimizu [13].)

Finally, we investigate decentralized markets, such as decentralized auction markets and decentralized markets with bargaining with periodic access to centralized auction markets. A question then arises as to whether such markets have determinate stationary equilibria or not. From the above discussion, such markets have a continuum of stationary equilibria, since the nature of monetary trades in both decentralized and centralized markets is of type (i). That is, the amount of money the sellers obtain is always equal to that of the buyers pay even out of equilibria, and thus the identity $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$ holds in all markets.

Remark 4 One may think that the indeterminacy result is due to the step value function. Along this line, Wallace and Zhu [16] present the notion of commodity-money refinement, and show that it eliminates all equilibria with step value functions. In our opinion, it is still an open question that some equilibrium refinement selects determinate equilibria. Kamiya and Shimizu [9] present a money search model where there is a continuum of stationary equilibria with strictly increasing value functions, and show that they survive the commodity-money refinement.

9 Conclusion

In this paper, we have developed a dynamic general equilibrium model with centralized auction markets, and showed that the outcomes are not necessarily Walrasian; the set of stationary equilibria in our model is a continuum that includes the Walrasian equilibrium. In addition, we built models on decentralized auction markets and on decentralized markets with bargaining, and obtained similar results. We also explored the logic behind the indeterminacy, and found that the equilibrium condition in auction market models has one degree of freedom; the number of equations is one less than that of the Walrasian market model. Note that search models with divisible money have a similar property. (See Kamiya and Shimizu [7].)

In the real world economy, most of the goods are traded in decentralized markets formulated as search models or in centralized markets without Walrasian auctioneers formulated as auction models. As we have shown, if the markets are formulated naturally, then the equilibrium condition has one degree of freedom. However, just one equilibrium is realized in the real world economy due to some factors not formulated in the model. It is our understanding that the factors slightly perturb the identity; if the government imposes some tax-subsidy rule, then the identity does not hold and equilibria become determinate, (see Kamiya and Shimizu [8]); a small number of goods might be traded in the Walrasian markets, and thus the identity does not hold; a small population growth rate can slightly perturb the identity. Even if the factors are much smaller than the volumes of trades and the inflow and outflow of the stationarity condition, they can change the equilibrium drastically. This is because there is one degree of freedom in the system of equilibrium equations without the factors. Thus, the equilibria crucially depend on the factors, even if they are small. In this sense, equilibria can be considered to be fragile.

Appendix

A Mixed Strategy Equilibria in Centralized Auction Markets

In this section, we prove the existence of mixed strategy stationary equilibria in the centralized auction markets model investigated in Section 3. A mixed strategy is defined as a function of a money holding $\eta \in R_+$ as follows:

$$\tilde{\xi} : R_+ \to \Omega\left((\{\sigma\} \times R_+) \cup (\{\beta\} \times R_+) \cup \{\nu\}\right),$$

where $\Omega(A)$ is the set of probability distributions on a set A.

We consider the following mixed (behavioral) strategy:

- an agent with $\eta \in [0, p)$ chooses to be a seller and posts an ask price p,
- an agent with p chooses to be a buyer and posts a bid price p with probability $\delta \in (0, 1)$, and chooses to be a seller and posts an ask price p with probability

$1-\delta$,

- an agent with $\eta \in (p, \infty)$ chooses to be a buyer and posts a bid price η.

Under the above strategy, money holding $2p$ is not a transient state. Let $h = (h_0, h_1, h_2)$ and

$$r = \frac{h_0 + (1-\delta)h_1 - h_2}{\delta h_1}. \tag{8}$$

An agent with p can buy a good with probability r. It is shown that $r \in [0,1]$ holds in equilibria. Under the above strategy, the stationarity of money holding distribution can be written as follows:

$$\begin{aligned}
&r\delta h_1 = h_0,\\
&h_2 + h_0 = r\delta h_1 + (1-\delta)h_1,\\
&h_2 = (1-\delta)h_1,\\
&h_0 + h_1 + h_2 = 1.
\end{aligned}$$

As we have shown in Section 8, two of the first three equations are redundant. The stationary distribution is obtained as follows:

$$h_0 = \frac{\delta r}{2-\delta+\delta r},\ h_1 = \frac{1}{2-\delta+\delta r},\ h_2 = \frac{1-\delta}{2-\delta+\delta r}. \tag{9}$$

Clearly, $r \geq 0$ implies $h_n \in [0,1]$ for $n = 0, 1, 2$.

If $\delta > 0$, all trades are made with a common price p. Thus, we obtain the Bellman equation as (1) in Section 3, where r is defined by (8). The incentive conditions for choosing the above strategy are as follows:

(i) $V(0) \geq 0$,

(ii) $V(p) = -c + \gamma V(2p)$.

The first inequality is the incentive for an agent who possesses no money to be a seller. Since $V(p)$ is the value when an agent becomes a buyer, the second equality implies that the agent is indifferent between being a buyer or being a seller. As in Section 3, the other incentive conditions can be easily checked. (i) is equivalent to (2) in Section 3, where r is defined by (8), and (ii) is equivalent to

$$r = \frac{\gamma\theta - 1}{(1+\gamma-\gamma^2)\theta - \gamma^2}. \tag{10}$$

Note that $\gamma > \frac{1}{\theta}$ implies $r \in (0,1)$. It is also worthwhile to note that, substituting (10) into (9) , (h_0, h_1, h_2) can be parameterized by δ. Next, substituting (10) into (2), we obtain the following inequality:

$$-(\theta+1)\gamma^3 + (\theta^2+\theta+1)\gamma^2 - (\theta-1)\gamma - \theta \geq 0.$$

It is verified that there exists $\underline{\gamma} \in \left(\frac{1}{\theta}, 1\right)$ such that the above inequality holds for any $\gamma \in \left(\underline{\gamma}, 1\right)$.

Theorem 6 There exists $\underline{\gamma} \in \left(\frac{1}{\theta}, 1\right)$ such that, for any given $\gamma \in \left(\underline{\gamma}, 1\right)$, the above mixed strategy is a stationary equilibrium with a discrete money holdings distribution for any $\delta \in (0, 1]$.

Note that since (h_0, h_1, h_2) and (V_0, V_1, V_2) depend on δ, there is real indeterminacy in equilibria.

B Relaxing the Participation Constraints

In this section, we relax the participation constraints and show that the similar results can be obtained as in Sections 4 and 3. We do so by assuming that agents in this section can simultaneously can participate in both transactions: sell and buy in the same period.

B.1 Walrasian Markets

We first consider Walrasian markets. Let the common price be $p \in R_+$. If the agent can afford to buy a good, i.e., $\eta \geq p$, then she can choose either ‘buy’, ‘sell’, ‘buy and sell’, or ‘do nothing’. Otherwise, the cash-in-advance constraint is binding and she can choose either ‘sell’ or ‘do nothing’. Thus, the Bellman equation is as follows:

$$V(\eta) = \begin{cases} \max\{u - c + \gamma V(\eta), u + \gamma V(\eta - p), -c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta \geq p, \\ \max\{-c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta < p. \end{cases}$$

For a given p, the unique value function $V : R_+ \to R$ and the optimal policy correspondence $\phi : R_+ \to \{\beta, \sigma, \omega, \nu\}$ are obtained, where β, σ , ω, and ν represent ‘buy’, ‘sell’, ‘buy and sell’, and ‘do nothing’. $\psi(\eta, x)$, the proportion of agents with η choosing $x \in \{\beta, \sigma, \omega, \nu\}$, F, a money holdings distribution, and T, a transition function, are defined as in Section 4.

As in the discussion in Section 4, an equilibrium price p, if it exists, is positive. Since an agent with $\eta \in [0, p)$ cannot buy,

$$V(\eta) = \max\{0, -c + \gamma V(\eta + p)\}$$

holds. Since an agent with $\eta + p$ can always buy and sell,

$$V(\eta + p) \geq \frac{u - c}{1 - \gamma},$$

holds. Thus, by

$$-c + \gamma \frac{u - c}{1 - \gamma} = \frac{\gamma u - c}{1 - \gamma} > 0,$$

we obtain

$$V(\eta) = -c + \gamma V(\eta + p), \tag{11}$$

i.e., $\phi(\eta) = \{\sigma\}$. Thus, by (11),

$$\begin{aligned} u - c + \gamma V(\eta) &= u - c + V(\eta - p) + c \\ &> u + \gamma V(\eta - p) \end{aligned}$$

holds for any $\eta \in [p, 2p)$. Since it is easily verified that $\sigma \notin \phi(\eta)$, we obtain $\phi(\eta) = \{\omega\}$. Moreover, it is also easily verified that $\phi(\eta) = \{\beta\}$ for any $\eta \in [2p, \infty)$. Then, we obtain the following result.

Theorem 7 There exists a stationary Walrasian equilibrium $\langle p, F, V, \phi, \psi \rangle$. Moreover, any Walrasian equilibrium is characterized by

(i) p and F satisfy

$$F\left([p,2p)\right)=1,$$
$$\int_p^{2p}\eta dF=M,$$

(ii) V is defined as

$$V(\eta)=\begin{cases}\frac{\gamma u-c}{1-\gamma}, & \text{if } \eta\in[0,p),\\ \frac{u-c}{1-\gamma}, & \text{if } \eta\in[p,2p),\\ \vdots & \\ \frac{1}{1-\gamma}\left\{u-\gamma^{n-1}c\right\}, & \text{if } \eta\in[np,(n+1)p),\\ \vdots & \end{cases}$$

(iii)

$$\phi(\eta)=\begin{cases}\{\sigma\}, & \text{if } \eta\in[0,p),\\ \{\omega\}, & \text{if } \eta\in[p,2p),\\ \{\beta\}, & \text{if } \eta\in[2p,\infty),\end{cases}$$

(iv)

$$\psi(\eta,x)=1,\quad \text{iff}\ \begin{cases}\eta\in[0,p) & \text{and}\quad x=\sigma,\quad \text{or}\\ \eta\in[p,2p) & \text{and}\quad x=\omega,\quad \text{or}\\ \eta\in[2p,\infty) & \text{and}\quad x=\beta.\end{cases}$$

Corollary 4 The robust stationary Walrasian equilibrium is unique.

B.2 Centralized Auction Markets

Next, we analyze centralized auction markets. We focus on stationary equilibria where the money holdings distribution has support $\{0,p\}$. We investigate the following strategy:

- an agent with $\eta\in[0,p)$ chooses to be just a seller and posts an ask price p,
- an agent with p chooses to be just a buyer and posts a bid price p,

- an agent with $\eta \in (p, 2p)$ chooses to be a buyer and a seller, posts an ask price p as a seller, and posts a bid price η as a buyer, and
- an agent with $\eta \in [2p, \infty)$ chooses to be just a buyer and posts a bid price η.

Under the above strategy, the Bellman equation is expressed as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta \in [0, p), \\ r\left(u + \gamma V(0)\right) + (1 - r)\gamma V(p), & \text{if } \eta = p, \\ u - c + \gamma V(\eta), & \text{if } \eta \in (p, 2p), \\ u + \gamma V(\eta - p), & \text{if } \eta \in [2p, \infty), \end{cases}$$

where

$$r = \frac{h_0}{1 - h_0}.$$

We decompose η into an integer multiple of p and a residual; that is, $\eta = np + \iota$, where n is a nonnegative integer and ι is a nonnegative real number less than p. Then, the value function becomes as follows:

$$V(np + \iota) = \begin{cases} \frac{r\gamma u - (1-\gamma+r\gamma)c}{(1-\gamma)(1+r\gamma)}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{1+r\gamma}\left[(1 - r + r\gamma)u + r\gamma c\right]\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{\gamma u - c}{1-\gamma}, & \text{if } \iota \neq 0,\ n = 0, \\ \frac{u - \gamma^{n-1}c}{1-\gamma}, & \text{if } \iota \neq 0,\ n \neq 0. \end{cases}$$

The incentive conditions are as follows:[10]

(i) incentive for an agent without money to be just a seller instead of doing nothing,

(ii) incentive for an agent with $\eta \in (0, p)$ to be just a seller instead of doing nothing,

(iii) incentive for an agent with p to be just a buyer instead of doing nothing,

(iv) incentive for an agent with p to be just a buyer instead of being just a seller,

(v) incentive for an agent with p to be just a buyer instead of being a buyer and seller,

(vi) incentive for an agent with $\eta \in (p, 2p)$ to be a buyer and seller instead of doing nothing,

---

[10] It is easily verified that an agent has no incentive to deviate by posting an ask (bid) price other than the equilibrium one.

(vii) incentive for an agent with $\eta \in (p, 2p)$ to be a buyer and seller instead of being just a seller,

(viii) incentive for an agent with $\eta \in (p, 2p)$ to be a buyer and seller instead of being just a buyer,

(ix) incentive for an agent with $\eta \geq 2p$ to be just a buyer instead of doing nothing,

(x) incentive for an agent with $\eta \geq 2p$ to be just a buyer instead of being just a seller, and

(xi) incentive for an agent with $\eta \geq 2p$ to be just a buyer instead of being a buyer and seller.

Among these, only (i), (iii), (v), and (xi) are relevant. It is easily verified that any other condition is automatically satisfied or reduced to one of the above four conditions.

First, since the value of $\eta = np$ is the same as the one in Section 3, (i) and (iii) are equivalent to (2) and (3) respectively.

Next, (v) is expressed as

$$V(p) \geq r\left[u - c + \gamma V(p)\right] + (1 - r)\left[-c + \gamma V(2p)\right],$$

and it is equivalent to

$$-\left[\gamma(2-\gamma)\theta - \gamma^2\right] r^2 + \left[\gamma(2-\gamma)\theta - \gamma^2\right] r - (\gamma\theta - 1) \geq 0. \tag{12}$$

Substituting (2) with equality into the RHS of (12), we obtain

$$\frac{\gamma\theta - 1}{\gamma(\theta - 1)^2}\left\{(\theta + 1)\gamma^2 - (\theta^2 + \theta + 2)\gamma + 2\theta\right\}.$$

Denote the expression in the braces by $F(\gamma)$, then we have

$$F\left(\frac{1}{\theta}\right) > 0,$$
$$F(1) < 0.$$

Therefore, there exists $\overline{\gamma}_1 \in \left(\frac{1}{\theta}, 1\right)$ such that for any $\gamma \in \left(\frac{1}{\theta}, \overline{\gamma}_1\right)$, there exists $\overline{r} \in \left(\frac{1-\gamma}{\gamma(\theta-1)}, 1\right]$ such that (2) and (12) hold for any $r \in [\underline{r}, \overline{r}]$, where $\underline{r} = \frac{1-\gamma}{\gamma(\theta-1)}$. Moreover, we can choose $\overline{\gamma}_1$ sufficiently close to $\frac{1}{\theta}$ such that (3) holds for any $r \in [\underline{r}, \overline{r}]$.

Lastly, (xi) is expressed by

$$V(\eta) \geq u - c + \gamma V(\eta), \quad \eta \geq 2p.$$

However, it is reduced

$$V(2p) \geq u - c + \gamma V(2p),$$

and it is equivalent to

$$-r(\theta+1)\gamma^2 + [r(\theta+1) - \theta]\,\gamma + 1 \geq 0. \tag{13}$$

Denote the LHS by $G(\gamma)$. Then, it is verified that

$$\begin{aligned} & G\left(\frac{1}{\theta}\right) > 0, \quad \text{if } r > 0, \\ & G(1) < 0. \end{aligned}$$

Then, there exists $\overline{\gamma}_2 \in \left(\frac{1}{\theta}, 1\right)$ such that for any $\gamma \in \left(\frac{1}{\theta}, \overline{\gamma}_2\right)$, (13) holds for any $r > 0$.

Finally, let $\overline{\gamma} = \min\{\overline{\gamma}_1, \overline{\gamma}_2\}$. Then for any given $\gamma \in \left(\frac{1}{\theta}, \overline{\gamma}\right)$, there exist $\underline{r}$ and $\overline{r}$ satisfying $0 < \underline{r} < \overline{r} \leq 1$ such that there exists a stationary equilibrium for any $r \in [\underline{r}, \overline{r}]$. Since $r = \frac{h_0}{1-h_0}$, we obtain the following result:

Theorem 8 There exists $\overline{\gamma} \in \left(\frac{1}{\theta}, 1\right)$ such that for any $\gamma \in \left(\frac{1}{\theta}, \overline{\gamma}\right)$, there exist $\overline{h}_0$ and $\underline{h}_0$ satisfying $0 < \underline{h}_0 < \overline{h}_0 \leq \frac{1}{2}$ such that there exists a robust stationary Markov equilibrium for any $h_0 \in \left[\underline{h}_0, \overline{h}_0\right]$.

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