Dynamic Auction Markets with Fiat Money

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First Draft: June 2008
This Draft: September 2009

Motivation

- In search models with divisible money, Green and Zhou (1998) show that there is a continuum of stationary equilibria with different real allocations (Real indeterminacy)
- Zhou (1999), Matsui and Shimizu (2005), Kamiya and Shimizu (2006)

Motivation (cont'd)

- Whether or not the real indeterminacy of stationary equilibria is an intrinsic property of random matching models?
- Kamiya and Shimizu (2006) show that it is an intrinsic property in some class of random matching models with divisible money
  - We show that if there exists a certain type of stationary equilibrium, generically there also exists a continuum of stationary equilibria under some regularity conditions
  - We present a general logic behind the real indeterminacy result

Motivation (cont'd)

- Lagos and Wright (2005): Access to Walrasian markets leads to the determinacy
- Does the real indeterminacy occur in a centralized market?
- This paper's answer: It can happen in a centralized market
- The real indeterminacy is not specific to decentralized transaction, but it is more general property of the economy with fiat money
- We also clarify why tradings in Walrasian markets make stationary equilibria determinate

Results

- We present an environment with fiat money
- Comparison of two trading mechanisms: **centralized auction markets** and **Walrasian markets**
- Centralized auction markets ⇒
  Real indeterminacy of stationary equilibria
- Walrasian markets (w/o and w/ lotteries) ⇒
  Real determinacy of stationary equilibria
- Intuition
- We also show the indeterminacy in decentralized auction markets and random matching markets

Time, Agents, and Goods

- Time is discrete and continues forever
- A continuum of infinitely lived agents with measure $\mathbf{1}$
- No double coincidence of wants
- ℓ types of agents with equal fraction
- ℓ types of goods: non-durable and indivisible
- Type $\mathbf{i}$ agent can produce one unit of type $\mathbf{i+1}$ good with cost $\mathbf{c > 0}$
- Type $\mathbf{i}$ agent obtains $\mathbf{u > 0}$ only if she consumes one unit of type $\mathbf{i}$ good

Money and Payoff

- Fiat money: durable and divisible
- $\mathbf{M} > \mathbf{0}$: the nominal stock of fiat money (fixed)
- $\boldsymbol{\theta} = \mathbf{u}/\mathbf{c} > \mathbf{1}$
- $\gamma \in (\mathbf{0}, \mathbf{1})$: discount factor
- Assumption: $\gamma > \mathbf{1}/\boldsymbol{\theta} \Leftrightarrow \gamma\mathbf{u} > \mathbf{c}$

Constraints and Refinement

- Cash-in-advance constraint
- Participation constraint:
  each agent chooses whether to be a seller, be a buyer, or do nothing in the beginning of each period
- Robustness to the money-holding-cost refinement:
  holding money incurs infinitesimally small cost
  $\Rightarrow$ no money hoarding

Centralized Auction Markets

- ℓ centralized auction markets are open in each period
- In each market a multi-person **k**-double auction is held for the exchange of each type of goods for money
  - Sellers post ask prices
  - Buyers post bid prices
  - We assume a buyer cannot post a bid price beyond her current money holding

k-Double Auction in the Case of Finite Agents

$ 150
B1

$ 100
B2

$ 70
B3

S1
$ 20

S2
$ 50

k-Double Auction in the Case of Finite Agents

$ 150 B1	$ 100 B2	$ 70 B3
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Price $= [\mathbf{k} \cdot \mathbf{100} + (\mathbf{1} - \mathbf{k}) \cdot \mathbf{50}]$

S1 $ 20	S2 $ 50

Candidate for Equilibrium

- All trades occur with $\mathbf{p} > \mathbf{0}$
- $\mathbf{h_n}$: measure of agents with $\mathbf{np}$ amount of money
- $\mathbf{h} = (\mathbf{h_0}, \mathbf{h_1})$: money holdings distribution
- An agent with $\eta \in [\mathbf{0}, \mathbf{p})$ chooses to be a seller and posts an ask price $\mathbf{p}$
- An agent with $\eta \in [\mathbf{p}, \infty)$ chooses to be a buyer and posts a bid price η

Case of $\mathbf{h_0} \leq \mathbf{1/2}$

- # Sellers $\leq$ # Buyers
- $\mathbf{r} = \frac{\mathbf{h_0}}{\mathbf{h_1}} \in (\mathbf{0}, \mathbf{1})$: probability that a buyer with **p** moeny can buy a good in the auction
- Stationary condition at $\mathbf{0}$: $\mathbf{h_1} \cdot \frac{\mathbf{h_0}}{\mathbf{h_1}} = \mathbf{h_0}$
- Stationary condition at **p**: $\mathbf{h_0} = \mathbf{h_1} \cdot \frac{\mathbf{h_0}}{\mathbf{h_1}}$
- These are automatically satisfied (identities)
- **p** is determined by $\mathbf{p h_1} = \mathbf{M}$

Value Function

$$\mathbf{V}(\eta) = \begin{cases} -\mathbf{c} + \gamma \mathbf{V}(\eta + \mathbf{p}) & \text{if } \eta < \mathbf{p} \\ \mathbf{r}\left(\mathbf{u} + \gamma \mathbf{V}(0)\right) + (1 - \mathbf{r})\gamma \mathbf{V}(\mathbf{p}) & \text{if } \eta = \mathbf{p} \\ \mathbf{u} + \gamma \mathbf{V}(\eta - \mathbf{p}) & \text{if } \eta > \mathbf{p} \end{cases}$$

$$\mathbf{V}(\mathbf{np} + \iota) =$$

$$\begin{cases} \frac{\mathbf{r}\gamma\mathbf{u} - (1-\gamma+\mathbf{r}\gamma)\mathbf{c}}{(1-\gamma)(1+\mathbf{r}\gamma)} & \text{if } \iota = 0,\ \mathbf{n} = 0 \\ \frac{1}{1-\gamma}\left\{\mathbf{u} - \frac{\gamma^{\mathbf{n}-1}}{1+\mathbf{r}\gamma}\left[(1 - \mathbf{r} + \mathbf{r}\gamma)\mathbf{u} + \mathbf{r}\gamma\mathbf{c}\right]\right\} & \text{if } \iota = 0,\ \mathbf{n} \neq 0 \\ \frac{1}{1-\gamma}\left\{\mathbf{u} - \frac{\gamma^{\mathbf{n}}}{1+\gamma}(\mathbf{u} + \mathbf{c})\right\} & \text{if } \iota \neq 0 \end{cases}$$

Incentive Conditions

$$\text{(I) } \mathbf{V(0) \geq 0} \qquad \Leftrightarrow \mathbf{r} \geq \frac{1-\gamma}{\gamma(\theta-1)}$$

$$\text{(II) } \mathbf{V(p) \geq -c + \gamma V(2p)} \qquad \Leftrightarrow \mathbf{r} \geq \frac{\gamma\theta - 1}{(1+\gamma-\gamma^2)\theta - \gamma^2}$$

$$\Leftrightarrow \mathbf{h_0} \in \left[\max\left\{\frac{1-\gamma}{\gamma(\theta-2)+1}, \frac{\gamma\theta-1}{(1+2\gamma-\gamma^2)\theta - 1 - \gamma^2}\right\}, \frac{1}{2}\right]$$

Real Indeterminacy in Centralized Auction Markets

- **Theorem 1:** There exists a robust stationary equilibrium for any $\mathbf{h_0}$ s.t.

$$\mathbf{h_0} \in \left[\mathbf{max}\left\{\frac{\mathbf{1}-\gamma}{\gamma(\theta-\mathbf{2})+\mathbf{1}}, \frac{\gamma\theta-\mathbf{1}}{(\mathbf{1}+\mathbf{2}\gamma-\gamma^2)\theta-\mathbf{1}-\gamma^2}\right\}, \frac{\mathbf{1}}{\mathbf{2}}\right]$$

- **Real indeterminacy** of robust stationary equilibria

Walrasian Markets

- ℓ spot Walrasian markets are open in each period
- In each market one type of goods is exchanged for money
- We characterize the set of all robust symmetric stationary equilibria

Equilibrium Conditions

- Each agent maximizes her discounted sum of expected utility
- A money holding distribution **F** is stationary
- $\int \eta \mathbf{dF} = \mathbf{M}$
- Supply = Demand (# Sellers = # Buyers)

Value Function

Fix a common price $\mathbf{p} > \mathbf{0}$,

$$\mathbf{V}(\eta) = \begin{cases} \max\{\mathbf{u} + \gamma\mathbf{V}(\eta - \mathbf{p}), -\mathbf{c} + \gamma\mathbf{V}(\eta + \mathbf{p}), \gamma\mathbf{V}(\eta)\} & \text{if } \eta \geq \mathbf{p} \\ \max\{-\mathbf{c} + \gamma\mathbf{V}(\eta + \mathbf{p}), \gamma\mathbf{V}(\eta)\} & \text{if } \eta < \mathbf{p} \end{cases}$$

$\Rightarrow$ Step value function with step $\mathbf{p}$

Equilibrium Strategy, Money Holdings Distribution, and Price

- An agent with $\eta \in [\mathbf{p}, \infty)$ always chooses to be a buyer
- Any state $\eta \in [\mathbf{2p}, \infty)$ is a transient
- Since $\gamma \mathbf{u} > \mathbf{c}$, an agent with $\eta \in [\mathbf{0}, \mathbf{p})$ always chooses to be a seller
- Robustness implies $\mathbf{F}(\{\mathbf{0}\}) = \mathbf{F}(\{\mathbf{p}\}) = \mathbf{1/2}$
- $\frac{1}{2}\mathbf{p} = \mathbf{M} \Rightarrow \mathbf{p} = \mathbf{2M}$

Real Determinacy in Walrasian Markets

- **Theorem 2:** A robust symmetric stationary equilibrium is unique in Walrasian markets
- **Remark 1:** We obtain a similar result in Walrasian markets with lotteries where can clear even in the case of $\mathbf{h_0} \neq \mathbf{h_1}$
- **Remark 2:** Even without robustness, real allocation of stationary equilibria is unique

Comparison

- **Corollary 1:** The set of the real allocations in centralized auction markets does not coincide with that in Walrasian markets
- **Remark 3:** Robustness requirement is not crucial

Walrasian Markets with Lotteries

- Introduce a notion of lotteries
- It allows market-clearing even when # Sellers $\neq$ # Buyers

Lotteries for Buyers

- A lottery for buyers: $\boldsymbol{\ell_b = (p_b, \lambda_b)}$
- $\mathbf{p_b}$: price of lottery
- $\boldsymbol{\lambda_b}$: prob of obtaining one unit of the good
- When an agent buys $\mathbf{q_b}$ amount of lotteries, the prob of obtaining one unit of the good is $\boldsymbol{q_b \lambda_b}$
- Constraint: $\boldsymbol{q_b \lambda_b \leq 1}$

Lotteries for Sellers

- A lottery for sellers: $\ell_s = (p_s, \lambda_s)$
- p_s: price of lottery
- λ_s: prob of supplying one unit of the good
- When an agent sells q_s amount of lotteries, the prob of supplying one unit of the good is $q_s\lambda_s$
- Constraint: $q_s\lambda_s \leq 1$

Individual Optimaization Problem

$$\max_{q_b, q_s} q_b \lambda_b u - q_s \lambda_s c + \gamma V(\eta')$$

$$\text{s.t. } q_b, q_s \in \mathcal{R}_+$$

$$\min\{q_b, q_s\} = 0$$

$$q_b \lambda_b \leq 1$$

$$q_s \lambda_s \leq 1$$

$$\eta' = \eta - q_b p_b + q_s p_s$$

$$q_b p_b \leq \eta$$

Market Clearing Conditions

$$\lambda_b Q_b(\ell_b, \ell_s) = \lambda_s Q_s(\ell_b, \ell_s)$$
$$p_b Q_b(\ell_b, \ell_s) = p_s Q_s(\ell_b, \ell_s)$$
$$\max\{\lambda_b, \lambda_s\} = 1$$

where Q_b and Q_s are aggregate demand and supply

$$\Rightarrow \frac{\lambda_b}{\lambda_s} = \frac{p_b}{p_s}$$

Case where $1 = \lambda_b \geq \lambda_s$

$$q_b^*(\eta) = \begin{cases} 0 & \text{if } \eta < \bar{\eta}, \\ \min\left\{\frac{1}{\lambda_b}, \frac{\eta}{p_b}\right\} = \min\left\{1, \frac{\eta}{p_b}\right\} & \text{if } \eta > \bar{\eta}, \end{cases}$$

$$q_s^*(\eta) = \begin{cases} \frac{p_b - \eta}{p_s} & \text{if } \eta < \bar{\eta}, \\ 0 & \text{if } \eta > \bar{\eta}, \end{cases}$$

where $\bar{\eta} \in (0, p_b)$

$\Rightarrow$ Any state except $\{0\}$ and $\{p_b\}$ is transient

$\Rightarrow$ Statonarity condition requires $F(\{0\}) = F(\{p_b\}) = 1/2$

Case where $1 = \lambda_s \geq \lambda_b$

$$q_b^*(\eta) = \begin{cases} 0 & \text{if } \eta < \bar{\eta}, \\ \min\left\{\frac{1}{\lambda_b}, \frac{\eta}{p_b}\right\} = \min\left\{\frac{p_s}{p_b}, \frac{\eta}{p_b}\right\} & \text{if } \eta > \bar{\eta}, \end{cases}$$

$$q_s^*(\eta) = \begin{cases} \frac{p_s - \eta}{p_s} & \text{if } \eta < \bar{\eta}, \\ 0 & \text{if } \eta > \bar{\eta}, \end{cases}$$

where $\bar{\eta} \in (0, p_s)$

$\Rightarrow$ Any state except $\{0\}$ and $\{p_s\}$ is transient

$\Rightarrow$ Statonarity condition requires $F(\{0\}) = F(\{p_s\}) = 1/2$

Real Determinacy in Walrasian Markets with Lotteries

- In any stationary equilibrium there exists a unique $\mathbf{p} > \mathbf{0}$ such that $\mathbf{F(\{0\}) = F(\{p\}) = 1/2}$
- $\mathbf{\frac{1}{2}p = M \Rightarrow p = 2M}$
- **Theorem 5:** The real allocation of stationary equilibria in Walras markets with lotteries is unuqe and the same as that in Walrasian markets without lotteries

Equilibrium Conditions for Centralized Auction Markets

- $\mathbf{p = \frac{M}{h_1}}$
- $\mathbf{h_0 + h_1 = 1}$
- $\mathbf{h_1 \cdot \frac{h_0}{h_1} = h_0}$
- $\mathbf{h_0 = h_1 \cdot \frac{h_0}{h_1}}$
- Incentive conditions

Equilibrium Conditions for Centralized Auction Markets

- $\mathbf{p} = \frac{\mathbf{M}}{\mathbf{h}_1}$
- $\mathbf{h}_0 + \mathbf{h}_1 = \mathbf{1}$
-
-
-

Equilibrium Conditions for Walrasian Markets

- $\mathbf{p = \frac{M}{h_1}}$
- $\mathbf{h_0 + h_1 = 1}$
- $\mathbf{h_1 = h_0}$
- $\mathbf{h_1 T(p, \{0\}) = h_0 T(0, \{p\})}$
- $\mathbf{h_0 T(0, \{p\}) = h_1 T(p, \{0\})}$
- Incentive conditions
- ($\mathbf{T(\eta, A)}$: transition probability from η to A)

Equilibrium Conditions for Walrasian Markets

- $\mathbf{p = \frac{M}{h_1}}$
- $\mathbf{h_0 + h_1 = 1}$
- $\mathbf{h_1 = h_0}$
-
-
-
-

Basic Logic of Kamiya and Shimizu (2006)

- Random matching models
- Single price equilibria with $\mathbf{p}$
- $\mathbf{h_n}$: measure of agents with $\mathbf{np}$ amount of money
- $\mathbf{h = (h_0, h_1, \ldots, h_N)}$: money holdings distribution
- $\mathbf{I_n(h)}$: inflow into $\mathbf{np}$
- $\mathbf{O_n(h)}$: outflow from $\mathbf{np}$
- 2 Identities:
  1. $\sum_{n=0}^{N} I_n = \sum_{n=0}^{N} O_n$
  2. $\sum_{n=0}^{N} nI_n = \sum_{n=0}^{N} nO_n$

Interpretation of the 2nd Identity

- $\sum_{n=0}^{N} npI_n = \sum_{n=0}^{N} npO_n$
- LHS: the sum of the money held by the agents involved in transaction, measured **after** transaction
- RHS: the sum of the money held by the agents involved in transaction, measured **before** transaction

Stationary Condition

Stationary condition: $\mathbf{h_0, h_1, \ldots, h_N}$ satisfies

$\mathbf{I_0(h) = O_0(h)}$
$\mathbf{I_1(h) = O_1(h)}$
$\mathbf{I_2(h) = O_2(h)}$
$\mathbf{\ldots}$
$\mathbf{I_N(h) = O_N(h)}$
$\mathbf{\sum_{n=0}^{N} h_n = 1}$

$\mathbf{(N+1)}$ variables
$\mathbf{N}$ independent equations
$\Rightarrow$ 1 degree of freedom

Two Natures of Monetary Trades

- The amount of money the sellers obtain is always equal to that the buyers pay even off the equilibrium: Centralized auction markets, random matching markets
  $\Rightarrow$ the previous logic applies
  $\Rightarrow$ indeterminacy of stationary equilibria
- The amount of money the sellers obtain is not necessarily equal to that the buyers pay off the equilibrium: Walrasian markets
  $\Rightarrow$ one more equilibrium equation
  $\Rightarrow$ determinacy of stationary equilibria

Conclusion

- We present an environment with fiat money in which
  - centralized auction markets do not determinate stationary equilibria, but
  - Walrasian markets determinate stationary equilibria
- Transaction in centralized markets does not necessarily lead to the determinacy of stationary equilibria
- The determinacy result in Walrasian markets comes from the existence of additional equilibrium condition

Future Research

- What kinds of factors determinate the equilibrium?
    - Trading opportunity in Walrasian markets: Lagos and Wright (2005)
    - Income redistribution policy: Kamiya and Shimizu (2007)
- What results from the interaction between multiple determinants?

Decentralized Auction Markets

- Each seller opens a 2nd price sealed-bid auction with minimum bid
- Every buyer randomizes over visiting auctions with identical minimum bid with equal probability
- Matching friction caused by coordination failure (Peters 1991)

Candidate for Equilibrium

- $\mathbf{h} = (\mathbf{h_0}, \mathbf{h_1})$
- An agent with $\eta \in [\mathbf{0}, \mathbf{p})$ chooses to be a seller and posts a minimum bid $\mathbf{p}$
- An agent with $\mathbf{p}$ chooses to be a buyer and bids $\mathbf{p}$
- An agent with $\eta \in (\mathbf{p}, \infty)$ chooses to be a buyer and bids η if he finds the other participants in the auction, $\mathbf{p}$ otherwise
- $\mathbf{R} = \frac{\mathbf{h_1}}{\mathbf{h_0}} \geq \mathbf{1} \Leftrightarrow \#$ Sellers $\leq \#$ Buyers

Matching Friction

- α: probability that a seller succeeds to sell a good on the equilibrium path
- $\alpha/\mathbf{R}$: probability that a buyer succeeds to buy a good on the equilibrium path
- Suppose # Seller $= \mathbf{f}$ and # Buyer $= \mathbf{Rf}$ where $\mathbf{f}$ is finite
- $\boldsymbol{\alpha = 1 - \left(1 - \frac{1}{f}\right)^{Rf} \underset{f \to \infty}{\longrightarrow} 1 - e^{-R}}$

Real Indeterminacy in Decentralized Auction Markets

- **Theorem 3:** There exists $\underline{\gamma}$, $\overline{\gamma}$, and $\overline{\mathbf{h}}_\mathbf{0}$ s.t.
  - $\frac{1}{\theta} < \underline{\gamma} < \overline{\gamma} < 1$
  - $\overline{\mathbf{h}}_\mathbf{0} \in (0, \frac{1}{2})$
  - for any $\gamma \in \left(\underline{\gamma}, \overline{\gamma}\right)$, there exists a robust stationary equilibrium for any $\mathbf{h}_\mathbf{0} \in \left(\overline{\mathbf{h}}_\mathbf{0}, \frac{1}{2}\right]$
- Real indeterminacy of stationary equilibria

Random Matching Markets

- Every agent is randomly matched with another agent with probability μ in each period
- She observes the partner's type, but not his money holding
- Seller's take-it-leave-it offer

Candidate for Equilibrium

- $\mathbf{h} = (\mathbf{h_0}, \mathbf{h_1})$
- An agent with $\eta \in [\mathbf{0}, \mathbf{p})$ chooses to be a seller and offers a price $\mathbf{p}$
- An agent with $\eta \in [\mathbf{p}, \infty)$ chooses to be a buyer

Real Indeterminacy in Random Matching Markets

- **Theorem 4:** Let $\underline{\gamma}_{\mathbf{R}} = \frac{\mathbf{1}}{\mathbf{1}+\mu\ell^{-1}(\theta-\mathbf{1})} > \frac{\mathbf{1}}{\theta}$. Then, for any $\gamma \in \left(\underline{\gamma}_{\mathbf{R}}, \mathbf{1}\right)$, there exists a robust stationary equilibrium for any $\mathbf{h_0} \in \left[\frac{\mathbf{1}-\gamma}{\gamma\mu\ell^{-1}(\theta-\mathbf{1})}, \mathbf{1}\right)$
- Real indeterminacy of stationary equilibria