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Stable Sunspot Equilibria with Private Information

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Abstract

This paper investigates whether suspot equilibria are stable under adaptive learning when there exists private information that makes agents' learning and their forecasts heterogeneous. Nakagawa (2015) shows that the existence about private information expands the regions of structural parameters allowing the economy to be learnable. Our paper incorporates such private information into the Evans and McGough (2005c)'s general reduced-form expectational model and examine how the learnability of sunspot equilibria are affected by the existence of private information.

Our main finding is that the existence of private information expands the parameter regions of learnable sunspot equilibria so that makes those equilibria empirically plausible. Although sunspot equilibria were not learnable without the strong negative expectational feedback in the literature, the equilibria under private information are able to be learnable with a positive expectational feedback. In addition, learnable sunspot equilibria are found to be empirically plausible in a standard NK model with private information.

JEL classification: C62; D83; E52

Keywords: Adaptive learning; Private information; Heterogeneous misspecification; Sunspot equiliria

1 Introduction

Stationary sunspot equilibria have been seen as a possible source for macroeconomic business cycle fluctuations that are caused by fluctuations in agents' expectations. Benhabib and Farmer (1994) and Farmer and Guo (1994) show that the economy can be indeterminate in RBC-type models with non-externalities and the dynamics of sunspot equilibria matched part of the US data of business cycle fluctuations. On the other hand, the stability of those equilibria under adaptive learning has not been fully clarified.

The learnability of sunspot equilibria was first demonstrated by Woodford (1990) in an overlapping generations model, but Evans and Honkapohja (2001) show that sunspot equilibria are unlikely to be learnable in a variety of RBC-type models. Evans and McGough (2005c) focus on the fact that sunspot equilibria may be represented as 'common factor' (CF) sunspot solutions, in which extraneous sunspot variables are autoregressive processes with resonance frequency coefficients, and argue that when agents form the forecasting models of CF representation, there exist learnable sunspot equilibria of CF representation in a general reduced-form expectational model. However, Evans and McGough (2005a) show that learnable CF sunspot equilibria are unlikely in RBC dynamic models because the strong negative expectational feedback is required for the learnability. Analytically, Duffy and Xiao (2007) argue that the learnability of CF sunspot equilibria is "empirically implausible" in the sense that the equilibria involve oscillatory adjustment that is not in line with RBC models with indeterminate REE and with observations in US aggregate data. To this situation, McGough, Meng, and Xue (2013) find the empirical plausibility of CF sunspot equilibria in non-convex RBC models, but the plausibility of those equilibria has not been fully investigated. In particular, few papers have not enough examined CF sunspot equilibria in indeterminate NK models, which have been increasingly focused upon as an explanation for recent business cycle fluctuations under zero lower bound (e.g., Belaygorod and Dueker, 2009; Hirose, 2013; Zheng and Guo, 2013). Berardi (2009) shows that the CF sunspot equilibria found by Evans and McGough (2005c) are stable under heterogeneous expectations, but the empirical plausibility of those equilibria are not confirmed.[1]

This paper investigates whether suspot equilibria are stable under adaptive learning when there exists private information that makes agents' learning and their forecasts heterogeneous. Nakagawa (2015) shows that the existence about private information expands the regions of structural parameters allowing the economy to be learnable. Our paper incorporates such private information into the Evans and McGough (2005c)'s general reduced-form expectational model and examine how the learnability of CF sunspot equilibria are affected by the existence of private information. In addition, this paper examines the learnability of sunspot equilibriain in a standard NK model with private information.

Our main finding is that the existence of private information expands the parameter regions of learnable CF sunspot equilibria so that makes those equilibria empirically plausible. Although sunspot

[1]SSEs of CF representation are also considered by Airaudo and Zanna (2010), Airaudo (2013), and Arifovic, Bullard, and Kostyshyna (2013), but their empirical plausibility are not discussed.

equilibria were not learnable without the strong negative expectational feedback in the literature, the equilibria under private information are able to be learnable with a positive expectational feedback. In addition, learnable CF sunspot equilibria are found to be empirically plausible in standard NK models with private information.

The paper is structured as follows. The next section presents our model and the E-stability conditions of CF sunspot equilibria. Section 3 introduces private information that makes agents' learning and forecasts heterogeneous. Section 4 calibrates the parameter regions of stable CF sunspot equilibria with private information and shows the regions of nominal interest rate rules that makes CF sunspot equilibria stable. Finally, we present our conclusions.

2 Model

2.1 Setup

We start with a version of the Evans and McGough (2005c)'s model of the general form of the univariate linear expectations model. The economy is represented by two vector equations:

$$y_t = \beta E_t^* y_{t+1} + \delta y_{t-1} + \gamma' v_t, \quad (1)$$

$$v_t = \Phi v_{t-1} + e_t. \quad (2)$$

The equations represent the dynamics of the endogenous variables and the evolution of the exogenous variables. The economy has one endogenous variables and n exogenous variables. $y_t \in \mathbb{R}$ is an endogenous variable at time t. $v_t = (v_{1t}, \ldots, v_{nt})' \in \mathbb{R}^n$ is a vector of autoregressive exogenous variables. The standard deviation of v_{it} for each i is defined by $\sigma_{ii} > 0$, and the correlation matrix of v_t is defined by $\Gamma \equiv \left(\rho_{ij}\right)_{1 \le i,j \le n}$, where $\rho_{ij} = \rho_{ji}$ and $\rho_{ij} \in [0, 1]$ denotes the correlation between v_i and v_j for each $i, j \in \{1, \ldots, n\}$. $e_t \in \mathbb{R}^n$ is a vector of fundamental shocks with means of zero that drive the stochastic process of v_t.[2] $\beta \in \mathbb{R}$ is a coefficient of $E_t^* y_{t+1}$, and $\delta \in \mathbb{R}$ is a coefficient of y_{t-1}, and $\gamma = (\gamma_1, ..., \gamma_n)' \in \mathbb{R}^n$, and $\Phi \in \mathbb{R}^n \times \mathbb{R}^n$ is a matrix of coefficients of v_t. E_t^* is the operator of the aggregate expectation of y_{t+1} at time t, which is not necessarily rational under adaptive learning.

For ease of calculation, we impose regularity assumptions on these parameters in Appendix A. In particular, Φ is assumed to be a diagonal and nonnegative matrix whose diagonal elements exist in the interval $[0, 1)$: $\Phi \equiv diag\left(\varphi_i\right)_{1 \le i \le n}$ where $0 \le \varphi_i < 1$ for each i. In addition, Γ is assumed to be a nonnegative matrix, in which $0 \le \rho_{ij} \le 1$ for each $i, j \in \{1, ..., n\}$. These assumptions are not crucial for our analysis, as most stationary linear models in the literature can be transformed to satisfy these conditions.

The original Evans and McGough (2005c)'s model is extended in two ways. First, we allow the existence of multiple exogenous variable ($n \ge 1$) in order to allow the existence of private information

[2] Note that exogenous variables with nonzero and heterogeneous means can be transformed to the form (2).

of those variables across agents. Second, we assume the AR(1) processes of exogenous variables (that is, $\varphi_i \neq 0$ for some i) in order to ensure the effect of the existence of private information on the stability of sunspot equilibria.

2.2 Irregular Case of REEs

First, to make it easier later to highlight the impact of private information, we review the Evans and McGough (2005c)'s results about the stability conditions of sunspot equilibria under learning with a full information set.

Under rational expectations, whether the system has a non-explosive solution depends upon the roots of the associated quadratic $\beta a^2 - a + \delta$. If a_1 and a_2 denote those roots, then

$$a_1 = \frac{1 - \sqrt{1 - 4\beta\delta}}{2\beta} \text{ and } a_2 = \frac{1 + \sqrt{1 - 4\beta\delta}}{2\beta}.$$

First, if $|a_1|, |a_2| > 1$, then almost surely there exist no non-explosive solutions to the system. Second, as a regular case, if $|a_1| < 1 < |a_2|$ (that is, $|\beta| < 1$), then there exists a unique non-explosive solution, which has the representation:

$$y_t = a_1 y_{t-1} + (\beta a_2)^{-1} \gamma' v_t,$$

which is stationary, in doubly infinite case, and asymptotically stationary for the initialized model. It follows that there do not exist SSEs. Finally, as an irregular case, if $|a_1|, |a_2| < 1$ (that is, $|\beta| > 1$), then there exist SSEs. In this case, y_t is a solution if and only if it can be written in the form

$$y_{t+1} = \beta^{-1} y_t - \beta^{-1} \delta y_{t-1} - \beta^{-1} \gamma' v_t + \varepsilon_{t+1},$$

for some martingale difference sequence $\varepsilon_{t+1} \equiv y_{t+1} - E_t y_{t+1}$ representing for agents' forecast error and $E_t \varepsilon_{t+1} = 0$ implied by rational expectations, and in this case every solution to the initialized model is indeterminate. In the irregular case, SSEs may have the representations of two types: a general form (GF) representaion,

$$y_t = \beta^{-1} y_{t-1} - \beta^{-1} \delta y_{t-2} - \beta^{-1} \gamma' v_{t-1} + \varepsilon_t,$$

and if the roots of the associated quadratic are real, a common factor (CF) representation,

$$\begin{aligned} y_t &= a_i y_{t-1} + \xi_t + (\beta a_j)^{-1} \gamma' v_t, \quad \text{for } i = 1, 2, \\ \xi_t &= a_j \xi_{t-1} - (\beta a_j)^{-1} \gamma' v_t + \varepsilon_t, \end{aligned} \tag{3}$$

where $a_i \in \mathbb{R}$ for $i = 1, 2$, and $\xi_t = \lambda \xi_{t-1} + \varepsilon_t$. If we denote $\tilde{\varepsilon}_t \equiv -(\beta a_j)^{-1} v_t + \varepsilon_t$, then $\xi_t = a_j \xi_{t-1} + \tilde{\varepsilon}_t$.

Finally, the region of the parameter space in which the model is irregular and SSEs exist: 1) $\beta > \frac{1}{2}, \delta > 1 - \beta, \delta < \beta$, and 2) $\beta < -\frac{1}{2}, \delta < -1 - \beta, \delta > \beta$.

2.3 Stability of Stable Sunspot Equilibria

If agents do not have enough knowledge to develop rational expectations, agents might adopt adaptive learning using all available data to formulate their forecast $E_t^* y_{t+1}$. In line with the representations of sunspot REEs obtained in the last section, we can consider PLMs of the form:

$$y_t = c + a y_{t-1} + b y_{t-2} + d\xi_t + k' v_t + l' v_{t-1}, \tag{4}$$

which captures the GF solutions (provided $\lambda = 0$) and the CF solutions (provided $\lambda = a_j$). We set $\theta = (a, b, c, d, k', l')$.

If we assume that agents have the information set of $\{y_s, v_s, \xi_s\}_{s=1}^t$, which is equivalent under rational expectations so that y_t and $E_t^* y_{t+1}$ are simultaneously determined at time t, agents' expectations are formed:[3]

$$E_t^* y_{t+1} = c + a y_t + b y_{t-1} + d\lambda\xi_t + \left(k'\Phi + l'\right) v_t.$$

This forecast is incorporated into Eq. (1) and yields the actual law of motion (hereafter, ALM) of the economy,

$$y_t = \frac{\beta}{1-\beta a} c + \frac{\beta b + \delta}{1-\beta a} y_{t-1} + \frac{\beta\lambda}{1-\beta a} d\xi_t + \frac{\beta\left(k'\Phi + l'\right) + \gamma'}{1-\beta a} v_t. \tag{5}$$

Evans and Honkapohja (2001, chapter 6) show that the convergence of (a, c) to the REE in real-time learning through least-squares techniques is governed by the associated ordinary differential equation (hereafter, ODE):

$$\frac{d}{d\tau}(\theta) = T(\theta) - \theta,$$

where $T(\theta) \equiv \left(\frac{\beta}{1-\beta a}, \frac{\delta}{1-\beta a}, \frac{\beta}{1-\beta a} d\lambda, \frac{\beta k'\Phi + \gamma'}{1-\beta a}\right)$ is the mapping from the PLM θ to the ALM $T(\theta)$, and τ denotes notional time. If the ODE is locally asymptotically stable and if $d \to\neq 0$, then θ locally converges to those of the sunspot equilibria, meaning that the SSE is found to be learnable under adaptive learning.

The T-maps in terms of b and l are given by

$$\begin{aligned} b &\to 0, \\ l &\to 0. \end{aligned}$$

These T-maps indicate that although the functional form of Eq. (4) captures both the GF and the CF representations, there exists no fixed point of the form of GF representations. Tho other T-maps are given by

[3] Forming $E_t^* y_{t+1}$ using an information set including current endogenous variables is considered by Honkapohja and Mitra (2006), McCallum (2007), Ellison and Pearlman (2011), and Bullard and Eusepi (2014).

$$
\begin{aligned}
c &\rightarrow \frac{\beta}{1-\beta a}c, \\
a &\rightarrow \frac{\delta}{1-\beta a}, \\
d &\rightarrow \frac{\beta\lambda}{1-\beta a}d, \\
k &\rightarrow \frac{\beta\Phi k+\gamma}{1-\beta a},
\end{aligned}
$$

which provides the fixed points of CF representation. Now we denote the set of fixed points: $S = \left\{(\theta^*, \lambda) \in \mathbb{R}^{5+2n} | T(\theta^*) = \theta^*\right\}$. Then, $S = S_{M,1} \cup S_{M,2} \cup S_{CF,1} \cup S_{CF,2}$, where for $i = 1, 2$,

$$
\begin{aligned}
S_{M,i} &= \left\{(\theta, \lambda) \in S | a = a_i, b = c = 0, d = 0, k = \left[(1-\beta a_i) I_n - \beta\Phi\right]^{-1}\gamma, l = 0, \lambda \neq a_j\right\}, \\
S_{CF,i} &= \left\{(\theta, \lambda) \in S | a = a_i, b = c = 0, k = \left[(1-\beta a_i) I_n - \beta\Phi\right]^{-1}\gamma, l = 0, \lambda = a_j\right\},
\end{aligned}
$$

where d is arbitrary in $S_{CF,i}$ for $i = 1, 2$. Note that if $\lambda = a_j$, then $\frac{\beta\lambda}{1-\beta a_i} = \beta(a_1 + a_2) = 1$ for $i, j = 1, 2$, $i \neq j$. Therefore, the following proposition about the form of fixed points is obtained:

Proposition 1 *Let $X_t = [y_{t-1}, y_{t-2}, 1, \xi_t, v_t, v_{t-1}]'$ and assume that the roots a_1, a_2 are real. If $\theta \in S_{M,i}$ or $\theta \in S_{CF,i}$ then $y_t = \theta' X_t$ is a common factor representation. If y_t is an REE for $i = 1, 2$ there exists $\theta \in S_{M,i} \cup S_{CF,i}$ and martingale difference sequence ε_t so that $y_t = \theta' X_t$.*

E-stability conditions for the CF stationary sunspot equilibria $S_{CF,1}$ and $S_{CF,2}$ are obtained:

Proposition 2 *The conditions for E-stability of the CF solutions $S_{CF,1}$ and $S_{CF,2}$ are as follows:*

1. *The CF solution set $S_{CF,1}$ is E-stable when $\beta < -\frac{1}{2}$, $\beta\delta < \frac{1}{4}$, and $\beta + \delta < -1$.*

2. *The CF solution set $S_{CF,2}$ is E-stable when $\delta < 0$ and $\beta + \delta > 1$.*

The region in (β, δ) space where the CF solution SSEs are E-stable is described in Figure 1 corresponding to the Figure 1 in Evans and McGough (2005c). In the case of t information set, the stable regions for the CF solution $S_{CF,1}$ exist the second and third quadrants, which requires the strong negative expectational feedback $\beta < 0$ emphasized by Evans and McGough (2005a) to generate sunspot equilibria that are stable under learning. In addition, there is a stable region for the CF solution $S_{CF,2}$ in the fourth quadrant, where $\beta > 0$ is allowed for stability. However, $\delta < 0$ must be ensured for stability.[4]

[4] If current endogenous variables y_t are excluded from the information set held by agents, the condition for E-stability of the CF solution $S_{CF,1}$ is the same as in the above proposition, and the CF solution $S_{CF,2}$ is E-unstable (see Evans and McGough, 2005c).

3 Private Information

In what follows, we introduce private information about exogenous variables w_t, such that each agent has access to information on only a subset of those variables.[5] We follow the assumption about private information made by Nakagawa (2015).

Assumption 1 *For any $i \in \{1, ..., n\}$, the evolution of an exogenous variable $\{v_{is}\}_{s=1}^{t}$ is observable for the proportion $\frac{1}{n}$ of agents (hereafter, type i) and unobservable for agents of other types.*

For analytical tractability, the population of each type is assumed to be the same at $\frac{1}{n}$, but Nakagawa (2015) confirms that the distribution of the populations of different types are not important for our analysis. Assumption 1 implies that agents of type i recognize the stochastic characteristics of v_{it} and do not recognize the correlations $\left\{\rho_{ij} = \rho_{ji}\right\}_{j \neq i}^{n}$ and the quantity n of exogenous variables. In this situation, the agent of type i has the set of public and private information $\{y_s, v_{is}\}_{s=1}^{t}$, which is limited and different from the information sets held by other types in terms of $\{v_{it}\}_{i=1}^{n}$.[6] This type of private information describes a feature of idiosyncratic fundamental shocks at a microeconomic level: for example, a preference shock possessed by a household (see Allen and Gale, 2004) and the profitability of a borrower in a financial market (see Stiglitz and Weiss, 1981); those shocks are likely to continue to be observable only for specific agents.

Under Assumption 1, the number n and the correlations $\left\{\rho_{ij}\right\}_{i,j=1}^{n}$ of exogenous variables may be treated as the degrees of the limitation and homogeneity in information sets, respectively. For later discussions, we treat $\left\{1 - \rho_{ij}\right\}_{i,j=1}^{n}$ as the degree of the heterogeneity in information sets. First, if $n = 1$, the information set of each agent is reduced to the full information set in Section 2.3. The larger n is, the more limited each information set is relative to the full one. Thus, n represents not only the quantity of privately observable variables but also the degree of limitation of each information set. Next, if $\rho_{ij} = \rho_{ji} = 1$ for some types i and j (and hence, $\varphi_i = \varphi_j$ and $\frac{w_{it}}{\sigma_{ii}} = \frac{w_{jt}}{\sigma_{jj}}$), the information sets of both types are perfectly homogeneous as if both types observed the same variable. The smaller ρ_{ij} is, the more heterogeneous are both information sets. Thus, $\left\{1 - \rho_{ij}\right\}_{i,j=1}^{n}$ represent the degree of heterogeneity in information sets of different types.

3.1 Determinacy of REE with Private Information

To the best of our knowledge, the general solution technology for rational expectations models with the private information considered in Assumption 1 has not been provided. Meanwhile, some studies

[5] Heinemann (2009) establishes another framework of private information in which agents receive different private noisy signals about a single economic variable.

[6] Note that Marcet and Sargent (1989) consider the private information of not only exogenous variables but also endogenous variables. They also consider the existence of hidden state variables that are unobservable by all agents. Our assumption is designed to focus on clarifying the impact of private information on the learnability of an equilibrium.

provide the results suggesting that the determincy condition of a fundamental REE would be independent of the informational structure about exogenous variables. Pearlman et al. (1986) provide the solution to the rational expectations models with partial information, which is included as a case of information sets considered in Assumption 1.[7] The obtained rational expectations equilibrium solution is equivalent with the REE obtained in the same models with full information in terms of the dynamics of endogenous and exogenous variables, and the solution is only defferent from the latter REE in terms of the influence from exogenous fundamental shocks and measurement errors. This fact implies that the determinacy conditions of the REE with partial information are also equivalent with the conditions of the REE with full information. As another version, Pearlman (1986) provides the solution to rational expectations models with diverse information, which sustains for several periods and are revealed later so that agents have full information after those periods. The obtained solution has the same dynamic property as the Pearlman et al. (1986)'s solution mentioned above. Following these facts, our paper conjectures that rational expectations equilibrium of the system (1)-(2) with private information in Assumption 1 would provide the same determinacy conditions as in Section 2.2.

3.2 Stability of an Equilibrium with Private Information

The agent of type i with the information set $\{y_s, w_{is}\}_{s=1}^{t}$ is constrained to estimate a heterogeneously misspecified PLM:

$$y_t = a_i y_{t-1} + c_i + d_i \xi_t + k_i v_{it}, \tag{6}$$

which underparameterizes the MSV solution (3) and differs from the PLMs of other types. Parameters a_i and c_i are $m \times 1$ vectors of constant terms and coefficients, and ε_{it} is an $m \times 1$ vector of error terms that are perceived to be white noise. ξ_t is a sunspot that has a AR(1) process:

$$\xi_t = \lambda \xi_{t-1} + \varepsilon_t.$$

Here we assume that different types of agents have the same sunspot. The degrees of heterogeneity and misspecification of PLMs of different types are represented by correlations of exogenous variables $\{\rho_{ij}\}_{i,j=1}^{n}$ and the number of exogenous variables n, respectively.

The agent estimates the parameter matrix $\phi_i' \equiv (c_i, a_i, d_i, k_i)$ using a recursive least-squares (hereafter, RLS) method, but does not recognize the misspecification in the PLM (6) as v_{it} and ε_{it} are orthogonalized. Finally, using the PLM (6) and the estimated ϕ_i, the agent formulates the forecast $E_{it}^{*} y_{t+1}$ as

$$E_{it}^{*} y_{t+1} = a_i y_t + c_i + d_i \lambda \xi_t + k_i \varphi_i v_{it}, \tag{7}$$

where E_{it}^{*} is the operator of expectations formed by type i at time t.

[7] Note that Pearlman et al. (1986) also consider partial information about not only exogenous variables, but also endogenous variables, while our Assumption 1 considers only the partial information about exogenous variables. However, this difference does not change our results.

The aggregate forecast $E_t^* y_{t+1}$ in Eq. (1) is determined by the aggregation of the forecasts of all types of the form (7). For simplicity, let us assume that forecasts of different types $\{E_{it}^* y_{t+1}\}_{i=1}^n$ have equal contributions to the dynamics of the economy.[8] Then, the aggregate forecast $E_t^* y_{t+1}$ is formulated by

$$E_t^* y_{t+1} = a y_t + c + d\lambda\xi_t + k'\Phi v_t,$$

where E_t^* is the operator of the average of heterogeneous forecasts $E_t^* = \frac{1}{n}\sum_{i=1}^n E_{it}^*$, and $a \equiv \frac{1}{n}\sum_{i=1}^n a_i$ is the average of the constant term vectors for all types, and $k' \equiv \frac{1}{n}(k_1, ..., k_n)$ is an $1 \times n$ matrix that combines the coefficients $\{k_i\}_{i=1}^n$ of the PLMs of the form (6) for all types and multiplies them by $\frac{1}{n}$. $c \equiv \frac{1}{n}\sum_{i=1}^n c_i$, $d \equiv \frac{1}{n}\sum_{i=1}^n d_i$.

The ALM of the economy depends upon $E_t^* y_{t+1}$ in Eq. (7). Substituting Eq. (7) into the system (1)–(2), the ALM is determined by

$$y_t = \frac{\beta}{1-\beta a}c + \frac{\delta}{1-\beta a}y_{t-1} + \frac{\beta}{1-\beta a}d\lambda\xi_t + \frac{\beta k'\Phi + \gamma'}{1-\beta a}v_t. \tag{8}$$

After recursive estimations, if $d \to\neq 0$, then there exist stable sunspot equilibria in the presence of private information.

Eq. (8) has the same form as the ALM (5) under the full information set and includes the latter ALM as a special case of $n = 1$ or $\{\rho_{ij} = 1\}_{i,j=1}^n$ in Assumption 1. Nakagawa (2015) indicates that the number of private information n (and hence, the number of the types of heterogeneous agents) and the correlations of exogeneous variables $\{\rho_{ij}\}_{i,j=1}^n$ represent the degrees of limitation and heterogeneity of the information sets held by agents, respectively, and hence the degrees of misspecification and heterogeneity of the PLMs of the form (6).

The stability of the equilibrium attainable under HM learning is subject to whether the aggregate parameters $\phi' \equiv (c, a, d, k')$ converge to bounded values; the dynamics of ϕ is established by the real-time learning processes of agents of all types for $\{\phi_i\}_{i=1}^n$. Under HM learning, where the PLMs are underparameterized, the principle does not necessarily hold, and the convergence of $\{\phi_i\}_{i=1}^n$ is inferred from the stochastic recursive algorithms of $\{\phi_i\}_{i=1}^n$ formulated by the PLMs of the form (6) for all i and the ALM (8).

4 Calibrations

In this section, we examine the empirical plausibility of stable SSEs with private information in standard macroeconomic model. First, we use the Evans and McGough (2005b)'s model of the general form of the multivariate linear expectations model. We calibrate the parameter regions that ensures the stability of SSEs with private information the impact of private information. Next, we use a basic NK macroeconomic model. The NK model is a benchmark macroeconomic framework for establishing DSGE models and analyzing optimal monetary policy.

[8]Note that the proportions of contributions of the different forecasts do not affect our results (see Nakagawa, 2015).

4.1 Evans and McGough (2005b)'s Model

First, we follow the parameter settings adopted by Evans and McGough (2005b) and calibrate parameter regions of stable SSEs in the presence of private information. For structural parameters, we assume $\gamma = 1$ and $n = 10$. For simplicity of calculations, we assume exogenous variables with the same stochastic characteristics: $\rho_{ij} = 0$ and $\varphi_i = 0.9$ for all i, j, and

$$E\left(v_i v_j\right) = \omega_{ij} = \begin{cases} \omega \text{ for } i = j, \\ \rho\omega, 0 \leq \rho \leq 1 \text{ for } i \neq j. \end{cases},$$

where $\omega = 1$ and $E\left(\xi^2\right) = 1$. We assume constant gain of least squares learning as 0.01.

Figure 2 shows that stable parameter regions extend into the first quadrant ($\beta > 0, \delta > 0$). Therefore, sunspot equilibria can be learnable in the first quadrant ($\beta > 0, \delta > 0$). The intuition behind our result is that the existence of private information makes the evolution of y_{t-1} more informative in making forecasts. When agents do not have the full information of exogenous variables, the evolution of y_t tells something about the evolutions of unobservable exogenous variables. As a result, the fixed point of the coefficient a_i of y_{t-1} in the PLM of type i is increased by the existence of private information. This increase makes the sign of a part of the fixed point of the constant term $\frac{\beta}{1-\beta a}c$ in the ALM so that the sign of one of the stability conditions with respect to the constant c from $\beta < 0$ with private information to $\beta > 0$ with private information. This result lifts the empirical implausibility imposed on the literature.

4.2 NK Model

We consider a basic NK model with a monetary policy shock ϵ_t along the lines of Woodford (2003):

$$x_t = -\alpha\left(i_t - E_t^* \pi_{t+1}\right) + E_t^* x_{t+1}, \tag{9}$$

$$\pi_t = \kappa x_t + \beta\left\{\theta E_t^* \pi_{t+1} + (1-\theta)\pi_{t-1}\right\}. \tag{10}$$

The model has three endogenous variables: output gap x_t, inflation rate π_t, and nominal interest rate i_t. Eq. (9) is a log-linearized intertemporal Euler equation that is derived from the households' optimal choice of consumption. Eq. (10) is a forward-looking Phillips curve that is derived from the optimizing behavior of monopolistically competitive firms with Calvo price setting. $\alpha > 0$, $\kappa > 0$, and $0 \leq \beta < 1$ are assumed. The central bank adopts a Taylor-type contemporaneous data interest rate rule:

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \epsilon_t, \tag{11}$$

where ϕ_π and ϕ_x are the policy parameters controlled by the central bank, and they are assumed to be nonnegative, and ϵ_t is a monetary policy shock.

To consider the impact of private information in the NK model, Nakagawa (2015) assume that ϵ_t is the aggregation of idiosyncratic monetary policy shocks: $\epsilon_t \equiv \sum_{i=1}^n \epsilon_{it}$, which stem from, for example, idiosyncratic preference shocks held by policy board members. The shock ϵ_{it} for each i follows an AR(1) process: $\epsilon_{it} = \varphi_i \epsilon_{i,t-1} + v_{it}$, where $0 \leq \varphi_i < 1$ and the disturbance term v_{it} has a zero mean. The correlation of ϵ_{it} and ϵ_{jt} is $\rho_{ij} \geq 0$ for each i, j. Under CS learning, ϵ_{it} for all i is observable for all agents, whereas under HM learning, the shock is observable for $1/n$ of agents and unobservable for other agents. The aggregate forecasts $(E_t^* x_{t+1}, E_t^* \pi_{t+1})$ are the averages of the forecasts of all types $\{(E_{it}^* x_{t+1}, E_{it}^* \pi_{t+1})\}_{i=1}^n$.

Replacing the aggregate forecast with the average of heterogeneous forecasts in the NK model is justified by following the assumptions given by Branch and McGough (2009), who develop a version of the NK model extended to include two types of agents who are identical except in the way they form expectations. In our model, because of the absence of idiosyncratic shocks in the Euler equation and the Phillips curve, the optimal decision rules of different agents underlying the two equations are identical except in the way of forming forecasts. Therefore, our model with heterogeneous forecasts shares the same form as the NK model with homogeneous forecasts.[9]

4.2.1 Taylor Principle

Let us obtain conditions imposed on monetary policy parameters (ϕ_π, ϕ_x) in the interest rate rule (11) for the determinacy and learnability of the fundamental REE in this model. As a seminal fact, (see Bullard and Mitra, 2002) find that the both conditions are equivalent as the Taylor principle:

$$\kappa (\phi_\pi - 1) + \phi_x (1 - \beta) > 0.$$

That is, SSEs under rational expectations exists if (ϕ_π, ϕ_x) violate the principle.

4.2.2 Calibrations

Next, we calibrate the parameter domains of (ϕ_π, ϕ_x) satisfying the learnability conditions in the presence of private information. For simplicity of calculations, we assume exogenous variables with the same stochastic characteristics: $\rho_{ij} = \rho \in [0, 1)$, and $\varphi_i = \varphi \in [0, 1)$ for all i, j. For robustness, we consider the structural parameters following Woodford (2003): $\alpha = 1/0.157$, $\kappa = 0.024$, and $\beta = 0.99$. We consider the existence of 10 idiosyncratic shocks ($n = 10$). We set the autocorrelations φ of the shocks to be equal to 0.9 (Milani, 2008) and the number of hetorogeneous agents to be equal to $n = 10$. In addition, we set the coefficient λ of CF sunspot to be equal to 0.5. We set $\rho = 0$.

Figure 3 shows that if there exists no heterogeneity by private information ($n = 1$), sunspot equilibria under learning with $\lambda = 0.5$ never exist when $\phi_\pi > 1$ such that $d = 0$. On the other hand, if there exists some heterogeneity ($n > 1$), stationary sunspot equilibria exist there such that $d > 0$. This means that if agents specify the PLM that includes the CF sunspot in the presence of private information, CF sunspot equilbria are learnable in the parameters of the Taylor principle, and that

[9] See Branch and Evans (2011) and Muto (2011) following the same methodology.

the central bank should respond to π_t or x_t more stringent than the Taylor principle to ensure the fundamental equilibrium under adaptive learning with private information.

In addition, the existence of private information can be a source for plausible fluctuations of the economy. Our results are distinguish with Evans and McGough (2005b), which explore the possibility of existence of stable sunspot equilibria under full information using several variants of the New Keyensian Monetary model, which incorporates the forward and backward looking components so that includes Eqs. (9)–(10) as a special case. Then, Evans and McGough (2005b) analyze a number of policy rules, which also incorporates the forward and backward looking components so that includes Eq. (11) as a special case. As a result, they find there exist stable CF sunspot equilibria only when the central bank responds to the expected inflation $E_t\pi_{t+1}$ (policy rules named PR_3 and PR_4 in their paper). However, those sunspot equilibria are known to perform period-by-period oscillatory convergence to the steady state so that violates the empirical plausibility defined by Duffy and Xiao (2007).

5 Conclusions

This paper investigates whether suspot equilibria are stable under adaptive learning when there exists private information that makes agents' learning and their forecasts heterogeneous. Nakagawa (2015) shows that the existence about private information expands the regions of structural parameters allowing the economy to be learnable. Our paper incorporates such private information into the Evans and McGough (2005c)'s general reduced-form expectational model and examine how the learnability of CF sunspot equilibria are affected by the existence of private information.

Our main finding is that the existence of private information expands the parameter regions of learnable CF sunspot equilibria so that makes those equilibria empirically plausible. Although sunspot equilibria were not learnable without the strong negative expectational feedback in the literature, the equilibria under private information are able to be learnable with a positive expectational feedback. In addition, learnable CF sunspot equilibria are found to be empirically plausible in a standard NK model with private information.

Appendix

A Regularity Assumptions

Assumption 2

1. $\det(I_m - B) \neq 0$ *and* $\det(I_{mn} - \Phi \otimes B) \neq 0$.

2. Φ *is a diagonal and nonnegative matrix whose diagonal elements exist in the interval* $[0, 1)$.

3. Γ *is a nonnegative matrix, in which* $0 \leq \rho_{ij} \leq 1$ *for each* $i, j \in \{1, ..., n\}$*.*

Assumption 2.1 avoids the possibility that a nonexplosive fundamental REE could be indeterminate (see Honkapohja and Mitra, 2006, Proposition 1).

The diagonal representation of Φ in Assumption 2.2 simplifies the analysis by equating the eigenvalues of Φ with its diagonal elements existing in the interval $[0, 1)$. Note that this assumption is not crucial for our analysis, because even if Φ were originally nondiagonal, Eq. (2) could be transformed to an equation that includes a diagonal autoregressive matrix by premultiplying Eq. (2) by the $n \times n$ matrix formed from the eigenvectors of Φ. The diagonal elements in the interval $[0, 1)$ ensure the stationarity of w_t.

Neither is Assumption 2.3 crucial for our analysis because any linear model can be transformed to the system with $\Gamma \geq 0_{n\times n}$. For example, if any ρ_{ij} is negative in an original model, this negative correlation can be transformed to be positive by changing the sign of w_i (or w_j) and redefining the correlation between $-w_i$ and w_j as $\rho_{ij} \geq 0$. Applying this transformation to any negative correlation, the original model is transformed to the system with $\Gamma \geq 0_{n\times n}$.

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Figure 1

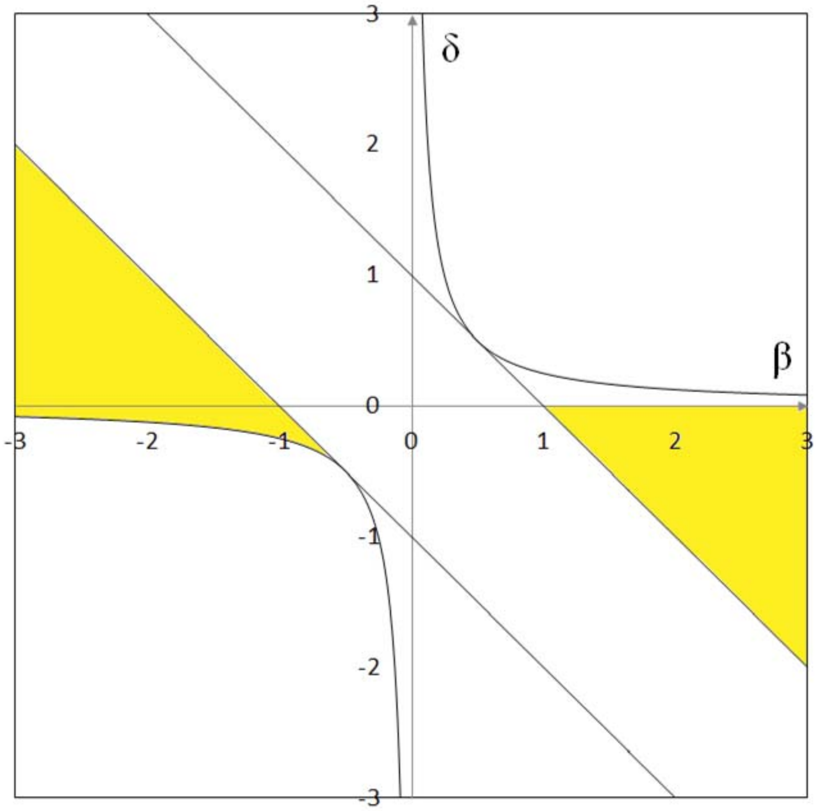

Figure 2

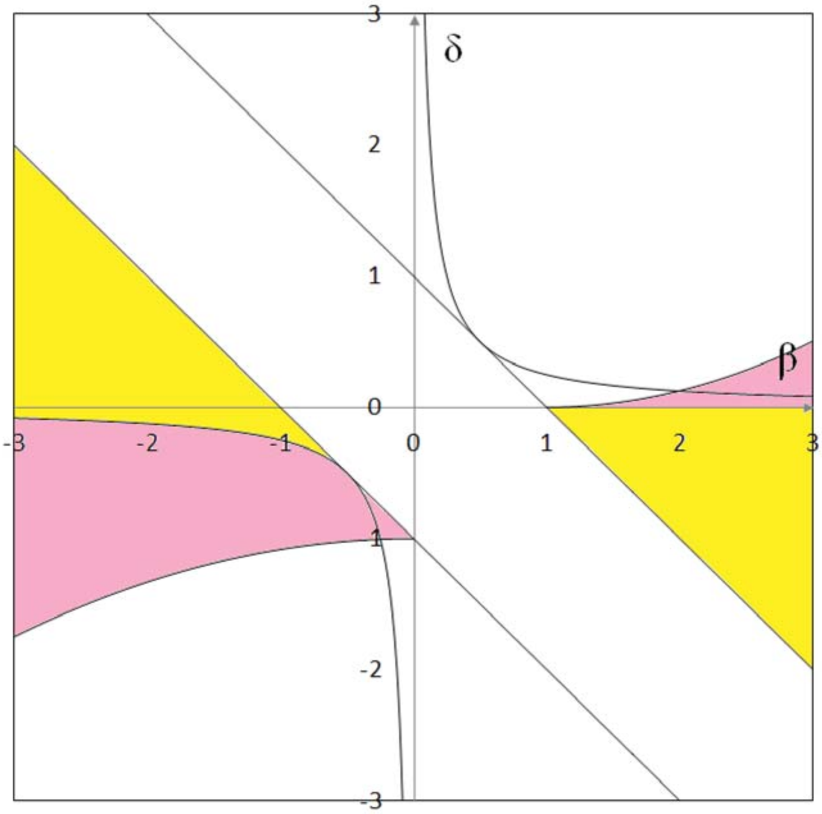

Figure 3